ON GENERALIZING THE

ENGLISH AUCTION

by

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The English auction for a single object possesses an extremely attractive property which is desirable to retain in multiple-object auctions. Suppose that the highest-value bidder's valuation equals v_1 and the second-highest-value bidder's valuation equals v_2. Then the "natural endpoint" for the English auction is for the highest-value bidder's bid to equal v_2 $(+\varepsilon)$, the second-highest-value bidder's bid to equal v_2, and thus for the highest-value bidder to win the object at a price of essentially the second-highest-value bidder's valuation. Or, to put things in more general terms, the natural endpoint for the English auction replicates the outcome of the Vickrey auction. This paper considers the circumstances under which the "natural endpoint" of a properly-specified generalization of the English auction replicates the outcome of the Vickrey auction for multiple objects, and begins to explore the implications of this.

The essential reason why it is desirable to replicate the outcome of the Vickrey auction is that: (a) the associated allocation is efficient; and (b) the associated pricing renders the mechanism strategy-proof. The fact that the allocation is efficient both makes the outcome socially desirable and means that post-auction resale is unnecessary. The fact that the Vickrey auction is strategy-proof makes it much more plausible that the "natural endpoint" of the auction will in fact be reached, as opposed to stopping prematurely at another (and, likely, inefficient) outcome.

This point can be made most clearly by considering auctions in which multiple objects are sold by setting a single independent price for each object (or type of object) and by accepting higher bids on every object until no new bids are submitted. A Walrasian equilibrium might be viewed as the "natural endpoint" of such an auction, since in a Walrasian equilibrium, there is a price for each object and no excess demand for any object. Moreover, it satisfies part (a) above, since the allocation under a Walrasian equilibrium is efficient. However, it does not satisfy part (b) since, unlike Vickrey-auction pricing, Walrasian pricing is not strategy-proof. As a consequence, auctions of this type are likely to lead bidders to strategically engage in "demand reduction," and the Walrasian outcome is unlikely to be reached. Rather, the outcomes are likely to be inefficient (Ausubel and Cramton, 1996, and Bolle, 1995).

Thus, ascending-bid auctions in which bidders submit independent bids for different objects are probably best viewed as false generalizations of the English auction. The approach of this paper is to instead consider auctions in which bidders submit bids for sets of objects. (In the literature, these are often referred to as "combinatorial auctions" or "package bidding.") At the end of the auction, the auctioneer accepts the n-tuple $\{(S_1,P_1), ..., (S_n,P_n)\}$ of compatible bids, one from each bidder i (i = 1,...,n), which maximizes revenues (i.e., the sum $P_1 + ... + P_n$). Each bidder i then wins the set S_i for a payment of P_i. This paper specifies an assumption on bidders' valuations under which Vickrey-auction pricing can be viewed as the "natural endpoint" of the combinatorial auction. Moreover, this assumption will sometimes be satisfied in environments where auctions which do not allow bids for sets of objects result in demand reduction. For example, any situation with multiple identical objects and where bidders exhibit diminishing marginal utilities satisfies these conditions, so the Vickrey-auction outcome is the natural endpoint. Thus, it might be argued that the combinatorial auction is a superior institution for this type of environment.

In many respects, this conclusion runs counter to the conventional wisdom on auction design. It is generally argued that auctions where bidders submit independent bids for each object are adequate for environments without complementarities between objects, and it is only when complementarities are present that package bids need be allowed.[1] By way of contrast, the conclusion here is that in the best-behaved environments (e.g., diminishing marginal utilities), auctions with package bids may offer clear efficiency advantages over auctions with only object-by-object bidding: the former are likely to culminate in Vickrey-auction pricing, while the latter are unlikely to end in Walrasian pricing due to demand reduction. In addition, there are some environments (where objects fail the standard requirement of being gross substitutes) in which package bidding is still likely to lead to Vickrey-auction pricing while a Walrasian equilibrium fails to exist. However, in environments with complementarities, both package bidding and object-by-object bidding are likely to fail to attain efficiency, so our present state of knowledge is probably then inadequate to rank the two varieties of auction format.

The two articles in the literature most closely related to the present paper are those of Bernheim and Whinston (1986) and Banks, Ledyard and Porter (1989). Bernheim and Whinston (1986) analyze the static combinatorial auction. In particular, they provide a rather neat characterizations of the set of coalition-proof Nash equilibria, which I borrow from heavily in my analysis below. They also emphasize that with two bidders, the outcome of the unique coalition-proof Nash equilibrium coincides with the Vickrey-auction outcome. Banks, Ledyard and Porter (1989) are the first authors (to my knowledge) who consider dynamic combinatorial auctions. They also hint at a connection between natural endpoints of dynamic combinatorial auctions and the Vickrey-auction outcome.

The present paper is also related to a number of articles which have examined the existence of Walrasian equilibrium in this discrete environment, and its implications. Kelso and Crawford (1982) examine Walrasian equilibrium in a matching context, establish the gross substitutes condition for existence, and demonstrate an adjustment process to equilibrium. Bikhchandani and Mamer (1997) and Gul and Stacchetti (1997a) further develop the theory of Walrasian equilibrium in multiple-object environments. Gul and Stacchetti (1997b) develop and analyze an ascending-bid auction procedure which achieves the Walrasian equilibrium. Ausubel (1997) proposes an ascending-clock auction for multiple identical objects which achieves the Vickrey-auction outcome under pure private values and outperforms the Vickrey auction under affiliated values. Milgrom (1997) analyzes and critiques some of the properties of the simultaneous ascending auction and the Banks-Ledyard-Porter AUSM procedure. Krishna and Perry (1997) develop some of the attractive properties of the Vickrey auction.

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[1]Bykowsky, Cull and Ledyard (1995), advocates of package bidding, state (p. 32): "In some environments a combinatorial auction is not needed. For example, the FCC recently used a simultaneous-independent auction to assign ten nationwide narrowband PCS licenses. This auction was characterized by a low degree of fitting complexity because of the absence of synergies among licenses. In such a bidding environment, a simple auction is the preferred auction form." Milgrom (1997), traditionally an advocate of independent bids, states (p. 16): "Some kind of combinatorial bidding is almost certainly beneficial in environments with significant complementarities." Ausubel, Cramton, McAfee and McMillan (1997) repeat this consensus view (p. 500): "Synergies, therefore, have major implications for the design of spectrum-license and other multiple-object auctions. If synergies are extreme, package bidding may be warranted in order to overcome the exposure problem. But if synergies are modest, auctions with package bidding offer little advantage, and given the complexity of these auctions, the simultaneous ascending auction is probably the more practical design."

1 Three Initial Examples

Auctions in which bidders place independent bids on different objects and the high bidders are awarded the respective objects — for example, the FCC's simultaneous ascending auction — do *not* appropriately generalize the English auction. Rather, outcomes of such auctions are closely related to the outcomes of the uniform price auction, and therefore exhibit a phenomenon known as "demand reduction" (Ausubel and Cramton, 1996). Large bidders in the auction have the incentive to reduce their demands on marginal units, in order to decrease the prices that they pay on inframarginal units. As a consequence, the auction exhibits inefficient outcomes — large bidders win too few objects, and small bidders win too many — in contrast to the efficiency afforded by the English auction.

This inefficiency is easily illustrated in the following two examples, each of which involve bidders who are presumed to possess complete information about both their own and their rivals' valuations. The first example, involving multiple identical objects, is taken from Ausubel (1997):

EXAMPLE 1. Suppose that there are five identical licenses for auction, and bidders have taste for up to three licenses. There are four bidders, whose marginal values are given as follows:

Bidder 1: $v_{1,1} = 123$, $v_{1,2} = 113$, $v_{1,3} = 103$

Bidder 2: $v_{2,1} = 75$, $v_{2,2} = 5$, $v_{2,3} = 3$

Bidder 3: $v_{3,1} = 125$, $v_{3,2} = 125$, $v_{3,3} = 49$

Bidder 4: $v_{4,1} = 85$, $v_{4,2} = 65$, $v_{4,3} = 7$

The above are marginal values for a first, second, and third license, respectively. For example, if Bidder 1 were to purchase two licenses at prices of 80 each, his total utility from the transaction would be computed by: $v_{1,1} + v_{1,2} - 80 - 80 = 123 + 113 - 160 = 76$.

The efficient outcome in Example 1 is easily seen to have Bidder 1 winning three units and Bidder 3 winning two units; and the efficient outcome is supported by a Walrasian price of 85 (or higher) per unit. However, this outcome should not be expected from an auction in which bidders place independent bids on the different objects. Instead, when the price first exceeds 75 — at which point Bidder 1 is demanding 3 units, Bidder 3 is demanding 2 units, and Bidder 4 is demanding 1 units — Bidder 1 recognizes that he has two available options. First, he can win 3 units by allowing the price to continue to rise to 85. Second, he can reduce his demand to just 2 units, thereby holding the price to essentially 75. Since the second option yields Bidder 1 a higher payoff than the first option (i.e., $123 + 113 - 2\times 75 > 123 + 113 + 103 - 3\times 85$), the auction yields an inefficient outcome: Bidder 4 necessarily wins one unit, and Bidder 1 only wins two.

Demand reduction is equally a phenomenon in auctions with dissimilar objects, as the second example, adapted from Ausubel (1996), illustrates.

EXAMPLE 2. Suppose that two dissimilar — but somewhat related — broadcast licenses, denoted A and B, are offered simultaneously for auction. Each bidder possesses a value for each license separately, and for the two licenses together. It is assumed that each bidder's values for these licenses are additively separable from their values for everything else in the world, that these values are expressible in monetary units, and that we normalize to zero the value associated with possessing neither license. There are two bidders, with values given as follows:

Bidder 1:

$$v_1(\emptyset) = 0$$
$$v_1(\{A\}) = 200$$
$$v_1(\{B\}) = 60$$
$$v_1(\{A,B\}) = 260 \;;$$

Bidder 2:

$$v_2(\emptyset) = 0$$
$$v_2(\{A\}) = 40$$
$$v_2(\{B\}) = 40$$
$$v_2(\{A,B\}) = 40 \;.$$

The efficient outcome in Example 2 is easily seen to have Bidder 1 winning both licenses; and the efficient outcome is supported by a Walrasian price of 40 (or higher) for each license. However, this outcome should not be expected in an auction in which bidders place independent bids on the two licenses. Instead, at the very outset, Bidder 1 recognizes that he has two available options. First, he can win both licenses by allowing the price on each to rise to 40. Second, he can reduce his demand to only license A, thereby holding the price to essentially zero. (So long as Bidder 2 expects to win license B, his marginal value attached to license A is zero.) Since the second option yields Bidder 1 a higher payoff than the first option (i.e., $200 - 1\times 0 > 260 - 2\times 40$), the auction yields an inefficient outcome: Bidder 2 wins license B, while Bidder 1 only wins license A.

Another problem which may arise in dissimilar-object auctions is that Walrasian prices may fail to exist — even when bidders' utilities exhibit diminishing marginal returns — as demonstrated by Kelso and Crawford (1982) and Gul and Stacchetti (1997a). The following example is borrowed from Gul and Stacchetti (1997a):

EXAMPLE 3. Suppose that there are two bidders for two apples and two bananas. Bidder 1 values a single unit of either fruit at \$1.00, two units of the same fruit at \$1.99, one unit of each fruit at \$1.50, and all sets of three or more fruits at \$1.99. Bidder 2's utility is additively separable in the two types of fruit: the first unit of either fruit is valued at \$1.00 and the second unit is valued at \$0.01.

If a Walrasian equilibrium exists, then it clearly prices the two apples at equal prices (p_a) and it clearly prices the two bananas at equal prices (p_b). Without loss of generality, suppose that $p_a \leq p_b$. Observe, however, that bidder 1 then can purchase two apples at no greater cost than one apple and one banana. Since he values the two apples strictly greater than one apple and one banana, he would never purchase one apple and one banana in a Walrasian equilibrium. At the same time, any Walrasian equilibrium must yield an efficient outcome, and the efficient outcome in this example is easily seen to have one apple and one banana allocated to each bidder. This allows us to conclude that no Walrasian equilibrium exists.

To conclude our discussion, we have seen that auctions in which bidders place independent bids on different objects appear to perform badly on each of Examples 1−3. We shall now examine an auction in which bidders are allowed to bid on packages of objects, and we shall see that these avoid inefficiency or nonexistence under some circumstances where the auction with independent bids misperform.

2 The Model

Let Ω denote the set of objects which are offered at auction, and let $N \equiv \{1, \ldots, n\}$ denote the set of n bidders. It is assumed that bidders' values for these objects are additively separable from their values for everything else in the world, and that these values are expressible in monetary units. Thus, each bidder i ($i = 1, \ldots, n$) possesses a finite value, $v_i(S)$, for every subset S of Ω, and we may normalize $v_i(\emptyset) = 0$. This paper will focus on the following auction procedures. Bidders submit bids (S,P), where S is a subset of the set of available objects and P is a positive real number. The winning bids are an n-tuple of compatible bids, one from each bidder, which maximizes the total revenues. Each winning bid is then accepted, meaning that if (S_i,P_i) is the bid from bidder i included in the winning bids, then bidder i receives the subset S_i and makes a payment of P_i; that is, the auction has the nature of a first-price auction. To be more precise:

DEFINITION 1. The *static combinatorial auction* is defined as the following static game:

(i) *Bids* consist of pairs (S,P), where $S \subset \Omega$ and $P \in \mathbb{R}_+$;

(ii) Each bidder i ($i = 1,\ldots,n$) simultaneously submits a finite collection $(S_i^1,P_i^1),\ldots,(S_i^K,P_i^K)$ of bids, where for each i, the various S_{ij} are required to be different subsets of Ω (i.e., $S_{ij} \neq S_{ik}$ for $j \neq k$);

(iii) In addition, the *zero bid* $(S_i^0,P_i^0) \equiv (\emptyset,0)$ is always taken as one of bidder i's bids;

(iv) The winning bids are determined by solving the problem of maximizing auction revenues: find an n-tuple $\{(S_1,P_1), \ldots, (S_n,P_n)\}$ of bids, one from each bidder i ($i = 1,\ldots,n$), which maximizes the sum $P_1 + \ldots + P_n$, subject to the constraint that the S_i are disjoint subsets of Ω;

(v) If the maximization problem of (iv) has a unique solution $\{(S_1,P_1), \ldots, (S_n,P_n)\}$, then each bidder i receives the subset S_i and makes the payment P_i;

(vi) If the maximization problem of (iv) has multiple solutions, then the auctioneer selects the solution which maximizes social surplus; and

(vii) If the maximization problem of (iv) has M solutions ($M \geq 2$) each of which maximize social surplus, then the auctioneer randomizes among them, assigning a probability of $1/M$ to each.

Let us also define an iterative version of the same auction:

DEFINITION 2. The *dynamic combinatorial auctions* is defined as the following dynamic game:

(a) In each period t of the game, steps (i)−(iv) of Definition 1 are followed, where in executing step (iv) in period t, the collection of bids evaluated is the union of all bids submitted in all periods $1, \ldots, t$;

(b) A *stopping rule* is maintained for determining the final period of the game (e.g., the current period is deemed the final period if the maximized revenue from the current period does not exceed the maximized revenue from the previous period by a predetermined amount);

(c) In the final period of the game, steps (v)−(vii) of Definition 1 are followed.

As a comparison with the static combinatorial auction (and as a way to characterize the outcome), it is useful to also define the familiar Vickrey auction (also often referred to as the Vickrey-Clarke-Groves mechanism; see Vickrey, 1961, Clarke, 1971, and Groves, 1973). Bidders submit bids (S,P), where S is a subset of the set of available objects and P is a positive real number. The winning bids are an n-tuple of compatible bids, one from each bidder, which maximizes the total revenues. Each bidder i ($i = 1,\ldots,n$) is then assigned the subset S_i of objects from his component of the winning bids, but his payment does not depend on P_i (instead based solely on the bids of other bidders); that is, the auction has the nature of a second-price auction. To be more precise:

DEFINITION 3. The *Vickrey auction* is defined as the following static game:

(i) *Bids* consist of pairs (S,P), where $S \subset \Omega$ and $P \in \mathbb{R}_+$;

(ii) Each bidder i ($i = 1,\ldots,n$) simultaneously submits a finite collection $(S_i^1,P_i^1), \ldots, (S_i^K,P_i^K)$ of bids, where for each i, the various S_{ij} are required to be different subsets of Ω (i.e., $S_{ij} \neq S_{ik}$ for $j \neq k$);

(iii) In addition, the *zero bid* $(S_i^0,P_i^0) \equiv (\varnothing,0)$ is always taken as one of bidder i's bids;

(iv) The winning bids are determined by solving the problem of maximizing auction revenues: find an n-tuple $\{(S_1,P_1), \ldots, (S_n,P_n)\}$ of bids, one from each bidder i ($i = 1,\ldots,n$), which maximizes the sum $P_1 + \ldots + P_n$, subject to the constraint that the S_i are disjoint subsets of Ω;

(v) For each bidder i ($i = 1,\ldots,n$), the auctioneer also computes an $(n-1)$-tuple of compatible bids, one from each bidder except bidder i, which maximizes the total revenues;

(vi) M is defined as the maximized total revenue from the maximization problem of (iv) and M_{-i} is defined as the maximized total revenue from the maximization problem of (v);

(vii) If the maximization problem of (iv) has a unique solution $\{(S_1,P_1), ..., (S_n,P_n)\}$, then each bidder i receives the subset S_i and makes the payment $P_i - M + M_{-i}$; and

(viii) If the maximization problem of (iv) has M solutions ($M \geq 2$), then the auctioneer randomizes among them, assigning a probability of $1/M$ to each.

Let us also define what is meant by an efficient allocation and the maximum attainable surplus:

DEFINITION 4. $\{E_i\}_{i \in I}$ is defined to be an *efficient allocation* and M_I is defined as the *maximum attainable surplus* (with respect to the subset $I \subset N$ of bidders) if $\{E_i\}_{i \in I}$ is a partition of Ω and:

(1) $$M_I \equiv \sum_{i \in I} v_i(E_i) \geq \sum_{i \in I} v_i(S_i),$$

for all partitions $\{S_i\}_{i \in I}$ of Ω.

The following proposition, which states that the Vickrey auction admits sincere bidding as a Nash equilibrium in dominant strategies, is well known:

THEOREM 0. If bidders have pure private values, then it is a Nash equilibrium in (weakly-) dominant strategies for each bidder i ($i = 1,...,n$) to submit the collection of bids $\{(S,v_i(S))\}_{S\subset\Omega}$, and such bidding results in an efficient allocation $\{E_1, \ldots, E_n\}$ along with payments of $v_i(E_i) - M_N + M_{N\setminus i}$.

Henceforth, when we refer to the "outcome of the Vickrey auction" or the "payments in the Vickrey auction," we will mean the outcome or payments associated with the sincere-bidding equilibrium of Theorem 0.

3 Preliminaries

Bernheim and Whinston (1986) perform a thorough analysis of the static combinatorial auction of Definition 1. They set up the following notation:

NOTATION 1. Let Π denote the set of all $x \equiv \{x_i\}_{i=1}^n \in \mathbb{R}^n$ which satisfy:

(2) $$\sum_{i \in I} x_i \leq M_N - M_{N \setminus I}, \text{ for all nonempty } I \subset N.$$

Further, let $\bar{\Pi}$ denote the Pareto frontier of Π, i.e., $\bar{\Pi} = \{ x \in \Pi \mid y \geq x \text{ is an element of } \Pi \text{ only if } y = x \}$.

The following result holds:

THEOREM 1 (Bernheim and Whinston, 1986, Theorem 3). In all coalition-proof Nash equilibria of the static combinatorial auction game, the allocation is efficient and bidders receive payoffs in $\bar{\Pi}$. Conversely, any payoff vector $x \in \bar{\Pi}$ can be supported by a coalition-proof Nash equilibrium.

Theorem 1 is also important to any consideration of the dynamic combinatorial auction game, for two reasons. First, we can utilize equilibrium strategies of the static game to define equilibrium strategies for the dynamic game, as follows. In the first period, submit exactly the bids from from a coalition-proof Nash equilibrium of the static game. In all subsequent periods, do not submit any additional bids. With appropriate specification of off-equilibrium strategies, it is easy to see that this forms a subgame-perfect equilibrium of the dynamic combinatorial auction game.

Second, if we wish to avoid a full consideration of the dynamic game, we can define a modified (and relatively simple) equilibrium notion for the static game which captures the idea that, if the equilibrium bids were all submitted before period T of the dynamic auction game, then no bidder (or coalition of bidders) would possess any incentive to submit any additional bids in period T. We define:

DEFINITION 5. Let $\sigma^* \equiv \{\sigma_i^*\}_{i \in N}$ denote a collection of existing bids for the set N of bidders, let $\sigma_I' \equiv \{\sigma_i'\}_{i \in I}$ denote a collection of additional bids for the nonempty subset I of bidders, and let $\tau_I \equiv \{\tau_i\}_{i \in I}$ (where $\tau_i = \sigma_i^* \cup \sigma_i'$) denote the union of existing and additional bids for bidders in I. Let $\{x_i\}_{i \in N}$ denote the payoffs associated with $(\sigma_I^*, \sigma_{-I}^*)$ and let $\{x_i'\}_{i \in N}$ denote the payoffs associated with (τ_I, σ_{-I}^*). σ^* will be said to be a *strong temporal equilibrium* of the static combinatorial auction game if it is always the case that the existence of $i \in I$ such that $x_i' > x_i$ implies the existence of $j \in I$ such that $x_j' < x_j$, i.e., if there is no coalition I who could all gain by deviating together to submit additional bids.

Observe that this definition (strong temporal equilibrium) differs from the standard definition (strong equilibrium) in only allowing bidders to deviate by submitting *additional* bids above and beyond their equilibrium bids. This is a natural requirement in a static game which attempts to be a collapsed version of a dynamic auction game. Given that the bidders have already submitted bids σ^*, we are asking the question whether bidders would have incentive to deviate. Since the existing bids have already been made — and withdrawals are not permitted — the only feasible deviations take the form of submitting additional bids.

Bernheim, Peleg and Whinston (1987, p. 3) remark that the requirements of strong Nash equilibrium are so stringent that they "almost never exist." In particular, it is easy to see that the static combinatorial auction does not admit any strong Nash equilibrium: immunity from deviations by the grand coalition requires all of the bids to be zero, but this allows smaller coalitions to profitably deviate. However, our modification of the definition, by restricting consideration to deviations of making additional bids, easily allows equilibria to exist. The only difference from the characterization of Theorem 1 is that Π replaces $\overline{\Pi}$. The proof now becomes a rather elementary version of the Bernheim-Whinston proof. Moreover, where the Bernheim-Whinston construction required bidders to submit menus of bids for all subsets $S \subset \Omega$, the current construction only requires a single bid by every bidder. We have:

THEOREM 2. In all strong temporal equilibria of the static combinatorial auction game, the allocation is efficient and bidders receive payoffs in Π. Conversely, any payoff vector $x \in \Pi$ can be supported by a strong temporal equilibrium.

PROOF. Observe that it is not feasible for inequality (2) to be violated for the grand coalition N. Suppose that the payoff vector $\{x_i\}_{i \in N}$ arising from players' bids violates inequality (2) for some $I \subsetneq N$. Let $J \equiv N \setminus I$. If R denotes the maximizing revenues from the existing bids, then $\sum_{j \in J} x_j < M_J - R$.

Define Δ to satisfy $0 < \Delta < [M_J - R - \sum_{j \in J} x_j]/|J|$ and set $x_j' = x_j + \Delta$ for all $j \in J$. Further, let $\{S_i'\}_{i \in N}$ be a partition of Ω which attains M_J in surplus for the bidders in J, and define $P_j' = v_j(S_j') - x_j'$ for all $j \in J$. Finally, let $J' \equiv \{j \in J \,|\, \text{bidder } j \text{ does not have a preexisting bid } (S_j', P_j'') \text{ such that } P_j'' > P_j'\}$. Observe that J' is nonempty, or else the maximized revenues from the existing bids would exceed R. Then if each bidder $j \in J'$ submits the additional bid (S_j', P_j'), these additional bids (in conjunction with the existing bids from $J \setminus J'$) win the auction and strictly improve the payoff of every bidder in J'. Hence, the payoffs $\{x_i\}_{i \in N}$ could not arise from a strong temporal equilibrium.

Suppose that the allocation arising from players' bids is inefficient. An analogous argument, in which the grand coalition deviates with bids which cause an efficient allocation to ensue, allows every bidder to strictly benefit. Again, this shows that an inefficient outcome could not arise from a strong temporal equilibrium.

Conversely, let $\{x_i\}_{i \in N}$ be any element of Π, let $\{E_i\}_{i \in N}$ be any efficient allocation, and define $P_i = v_i(E_i) - x_i$. We will now see that each bidder i submitting a bid of (E_i, P_i) forms a strong temporal equilibrium. Let $R = \sum_{i \in N} x_i$ denote the revenue from the existing bids. Suppose that any bidders were to deviate by submitting additional bids (S_j', P_j'), and suppose that the additional bids $\{S_j', P_j'\}_{j \in J'}$ were to combine with the existing bids $\{E_j, P_j\}_{j \in J''}$ (where $J' \cap J'' = \emptyset$) to yield revenues $R' > R$. Define $J = J' \cup J''$. Taking inequality (2) for $I \equiv N \setminus J$ yields $\sum_{i \in I} x_i \leq M_N - M_J$. Also note that $\sum_{i \in N} x_i = M_N - R$. Subtracting yields: $\sum_{j \in J} x_j \geq M_J - R$. Meanwhile, let $\{x_i'\}_{i \in N}$ denote the payoffs associated with the deviation. Observe that $\sum_{j \in J} x_j \leq M_J - R' < M_J - R$, and that by construction, $x_j' = x_j$ for all $j \in J''$. We conclude that there exists $j \in J'$ such that $x_j' < x_j$, allowing us to conclude from Definition 5 that each bidder i submitting a bid of (E_i, P_i) forms a strong temporal equilibrium. ■

4 Vickrey-Auction Pricing

We will now explore the relationship between the equilibrium outcomes of the static combinatorial auction developed in the previous section and the outcome of the Vickrey auction. We begin with a definition, associated notation, and assumption:

DEFINITION 6. For every $I \subset N$, we will define the *incremental surplus of I*, denoted $\mu(I)$, by:

$$\mu(I) = M_N - M_{N \setminus I}.$$

ASSUMPTION A1. The bidders' payoffs have the property that *the incremental surplus of the whole exceeds the sum of its parts* if:

$$\text{(A1)} \qquad \mu(I) \geq \sum_{i \in I} \mu(\{i\}), \text{ for all nonempty } I \subset N.$$

This assumption, as stated above, is not especially intuitive. Therefore, it is instructive to mention two somewhat stronger assumptions which are easier to interpret. First, one could instead assume that the incremental surplus function, $\mu(I)$, is superadditive. Second, one could instead assume that there are diminishing marginal returns *in bidders*: for every $I \subset J \subset N$ and $k \notin J$, we have

$M_{J \cup \{k\}} - M_J \leq M_{I \cup \{k\}} - M_I$. However, while these conditions are more readily interpretable, they each impose unnecessary constraints over and above Assumption A1. We have:

THEOREM 3. The set $\bar{\Pi}$ (from Notation 1) is a singleton if and only if Assumption A1 is satisfied. Moreover, in this event, $\bar{\Pi}$ coincides with the payoffs from the Vickrey auction.

PROOF. Suppose that Assumption A1 is satisfied, and consider any payoff vector $x \equiv \{x_i\}_{i=1}^n \in \mathbb{R}^n$ which satisfies $x_i \leq \mu(\{i\}) \equiv M_N - M_{N \setminus \{i\}}$ for all $i \in N$. Then, for any $I \subset N$ containing two or more elements, summing the aforementioned inequalities over all $i \in I$ yields $\sum_{i \in I} x_i \leq \sum_{i \in N} \mu(\{i\}) \leq \mu(I)$. Thus, inequality (2), for all $I \subset N$ containing two or more elements, is automatically satisfied. Clearly then, $x_i = \mu(\{i\}) \equiv M_N - M_{N \setminus \{i\}}$, for all $i \in N$, is the Pareto optimum of all such payoff vectors, and this $\{x_i\}_{i=1}^n$ equals the payoffs in the dominant-strategy equilibrium of the Vickrey auction.

Conversely, suppose that Assumption A1 is not satisfied. Then there exists $I \subset N$ such that $\sum_{i \in N} \mu(\{i\}) > \mu(I)$ and, consequently, inequality (2) for this I is not redundant with the n inequalities $x_i \leq \mu(\{i\}) \equiv M_N - M_{N \setminus \{i\}}$. Therefore, the Vickrey auction payoff $x_i = \mu(\{i\}) \equiv M_N - M_{N \setminus \{i\}}$, for all $i \in N$, is not an element of Π. It follows that one or more of the constraints forming the upper boundary of Π are *not* of the form $x_i \leq k_i$ for $i \in N$ and $k_i \in \mathbb{R}$, and therefore the Pareto frontier cannot be a singleto■

While Theorem 3 is not by itself easily applicable, it lends itself to two immediate corollaries. The first is due to Bernheim and Whinston (1986):

COROLLARY 1 (Bernheim and Whinston, 1986). In auctions with two bidders, Assumption A1 is automatically satisfied, and so the set $\bar{\Pi}$ is a singleton.

COROLLARY 2. In auctions of multiple identical objects with (weakly) diminishing marginal returns, Assumption A1 is satisfied and so the set $\bar{\Pi}$ is a singleton.

Observe that Corollaries 1 and 2 already treat the three examples with which we began the paper. Example 1 has multiple identical objects and diminishing returns; while Examples 2 and 3 each have only two bidders. (However, observe that Examples 2 and 3 do not crucially rely on there only being two bidders. For example, if Example 3 is augmented by adding a third bidder 2′ who is identical to bidder 2, and if a third apple and banana are introduced, nonexistence of Walrasian equilibrium, together with satisfaction of Assumption A1, are preserved.)

When Assumption A1 is satisfied, and so the set $\bar{\Pi}$ is a singleton, it seems fair to think of the Vickrey-auction outcome as the "natural endpoint" of the combinatorial auction. Any payoffs below the Vickrey-auction payoffs are unnecessarily low, to no bidder's advantage. Any payoffs above the Vickrey-auction payoffs can be defeated by additional bids from coalitions of bidders. Thus, it would seem that the Vickrey-auction payoffs give us a fairly compelling prediction as to the outcome of the combinatorial auction.

Assumption A1 can even be satisfied under circumstances where no Walrasian equilibrium exists (e.g., Example 3). And since bidders' payoffs are those from the Vickrey auction, there is no issue as to bidders strategically reducing their demands (e.g., Examples 1 and 2): as argued in Ausubel and Cramton (1996) and Bolle (1995), demand reduction is a phenomenon which derives from auction formats yielding prices different from those of the Vickrey auction.

Finally, one aspect of combinatorial auctions which has extensively been criticized is that they may give rise to a free-rider problem. For example, consider a situation where Bidder 1 holds the standing high bid on the doubleton {A,B}, but Bidders 2 and 3 together value the doubleton higher, e.g., $v_2(\{A\}) + v_3(\{B\}) > v_1(\{A,B\})$. Then, in general, it might be the case that Bidders 2 and 3 fail to displace Bidder 1's standing high bid, on account that each waits for the other to contribute the larger share toward displacing Bidder 1. However, when Assumption A1 is satisfied, there is *no* free-rider problem in generating these bids. Unless Bidder 1 has already bid so high that he will (unnecessarily) receive less than his Vickrey-auction payoff, no bidder in the displacing coalition need volunteer any more than will preserve his Vickrey-auction payoff. And no bidder in the displacing coalition need hesitate about volunteering that large a bid, since whenever the auction ends, he cannot expect any bid which would leave him more than his Vickrey-auction payoff to be sustained.

5 Conclusion

When bidders have the property that the incremental surplus of the whole exceeds the sum of its parts (Assumption A1), we conclude that the "natural endpoint" of the combinatorial auction yields an efficient outcome and Vickrey-auction pricing, whereas auctions in which bidders submit independent bids for different objects are unlikely to reach the efficient outcome associated with the Walrasian equilibrium, due to demand reduction. Moreover, under this assumption, a Walrasian equilibrium may fail even to exist, making efficiency in auctions in which bidders submit independent bids even less likely to occur.

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