An Efficient Ascending-Bid Auction for Multiple Objects

By LAWRENCE M. AUSUBEL*

When bidders exhibit multi-unit demands, standard auction methods generally yield inefficient outcomes. This article proposes a new ascending-bid auction for homogeneous goods, such as Treasury bills or telecommunications spectrum. The auctioneer announces a price and bidders respond with quantities. Items are awarded at the current price whenever they are "clinched," and the price is incremented until the market clears. With private values, this (dynamic) auction yields the same outcome as the (sealed-bid) Vickrey auction, but has advantages of simplicity and privacy preservation. With interdependent values, this auction may retain efficiency, whereas the Vickrey auction suffers from a generalized Winner's Curse. (*JEL* D44)

The auctions literature has provided us with two fundamental prescriptions guiding effective auction design. First, an auction should be structured so that the price paid by a player—conditional on winning—is as independent as possible of her own bids (William Vickrey, 1961). Ideally, the winner's price should depend solely on opposing participants' bids—as in the sealed-bid, second-price auction—so that each participant has full incentive to reveal truthfully her value for the good. Second, an auction should be structured in an open fashion that maximizes the information made available to each participant at the time she places her bids (Paul R. Milgrom and Robert J. Weber, 1982a). When bidders' signals are affiliated and there is a common-value component to valuation, an open ascending-bid format may induce participants to bid more aggressively (on average) than in a sealed-bid format, since participants can infer greater information about their opponents' signals at the time they place their final bids.

In single-item environments, these dual prescriptions are often taken to imply the desirability of the English auction and to explain its prevalence (see, for example, the excellent surveys of R. Preston McAfee and John McMillan, 1987; Milgrom, 1987). For auctions where bidders acquire multiple items, however, no one appears to have combined these two broad insights and taken them to their logical conclusion. The current article does precisely that: I propose a new ascending-bid auction format for multiple objects that literally takes heed of the two traditional prescriptions for auction design.[1]

* Department of Economics, University of Maryland, Tydings Hall, Room 3105, College Park, MD 20742 (e-mail: ausubel@econ.umd.edu; http://www.ausubel.com). I am extraordinarily grateful to Kathleen Ausubel and Peter Cramton for numerous helpful discussions. I also wish to thank Theodore Groves, Ronald Harstad, John Ledyard, Eric Maskin, Preston McAfee, Paul Milgrom, Philip Reny, Vernon Smith, Robert Wilson, and four anonymous referees for useful feedback, comments, suggestions, and explanations at various stages in the preparation of this article. I appreciate valuable comments from participants and discussants at the Princeton University Conference on the Spectrum Auctions, the Universitat Pompeu Fabra Conference on Auctions, the North American meetings of the Econometric Society, the American Economic Association meetings, the Utah Winter Finance Conference, and the NSF Decentralization Conference, as well as from seminar participants at numerous universities. Intellectual Property Disclosure: The auction design introduced in this article may be subject to issued or pending patents, in particular, US Patent No. 6,026,383 and US Patent Application No. 09/397,008. Upon request, the author will grant a royalty-free license to the referenced properties for noncommercial research on this auction design.

This article is dedicated to William Vickrey (1914–1996).

[1] Some readers may feel that my opening paragraphs overstate the case made in the literature for dynamic auction formats or, perhaps, the case for using auction mechanisms at all. Indeed, the revenue rankings favoring dynamic auctions depend critically on several strong assumptions, including affiliated random variables, symmetry, and risk neutrality. Moreover, when buyers have strongly interdependent values, some may consider auctions to be poor ways to generate revenues and may argue that other proce-

The starting point for understanding the design proposed herein is to consider the uniform-price auction. Recall that the classic English auction for a single object can be sensibly collapsed down to a sealed-bid, second-price auction. Analogously, most existing ascending-bid auction designs for identical objects can be sensibly collapsed down to the uniform-price auction, in which bidders simultaneously submit bids comprising demand curves, the auctioneer determines the clearing price, and all bids exceeding the clearing price are deemed winning bids at the clearing price. Unfortunately, the uniform-price auction possesses a continuum of equilibria yielding less than the competitive price (Robert Wilson, 1979; Kerry Back and Jaime F. Zender, 1993) and, indeed, with private information, *every* equilibrium of the uniform-price auction yields inefficient outcomes with positive probability (Ausubel and Peter Cramton, 2002). The reason for inefficiency is that uniform pricing creates strong incentives for "demand reduction": a bidder will bid less than her value for a marginal unit, in order to depress the price that she pays for inframarginal units.

More extreme results are possible if the auction is explicitly sequential. Ausubel and Jesse A. Schwartz (1999) show that a two-bidder, alternating-bid version of the uniform-price ascending auction possesses a unique subgame perfect equilibrium. The first bidder bids the opening price on slightly more than half the units, the second bidder bids the next possible price on the remaining units, and the game ends. Thus, the allocation need not bear any connection to an efficient outcome, and the price (one bid increment above the reserve price) need not bear any connection to a competitive price. While this prediction is admittedly extreme, it was almost perfectly borne out, empirically, in an October 1999 German spectrum auction.[2]

By way of contrast, the *(multi-unit) Vickrey auction* is an effective static design when bidders with pure private values have tastes for consuming more than one object. Again, bidders submit sealed bids comprising demand curves, the auctioneer determines the clearing price, and all bids exceeding the clearing price are deemed winning bids. The price paid for each unit won, however, is neither the amount of the bid nor the clearing price, but the *opportunity cost* of assigning this unit to the winning bidder. For discrete objects, if bidder i is to be awarded k objects, then she is charged the amount of the k^{th} highest rejected bid (other than her own) for her first unit, the $(k - 1)^{\text{st}}$ highest rejected bid (other than her own) for her second unit, ... , and the highest rejected bid (other than her own) for her k^{th} unit. For M divisible objects, Figure 1 depicts the outcome: $x_i(p)$ denotes bidder i's demand curve, $M - x_{-i}(p)$ denotes the residual supply after subtracting out the demands of all other bidders, p^* denotes the market-clearing price if all bidders participate in the auction, and p^*_{-i} denotes the market-clearing price in the absence of bidder i. The Vickrey auction awards a quantity of $x_i(p^*)$ to bidder i, and requires a payment equal to the area of the shaded region in Figure 1. Thus, each participant's payment (conditional upon winning a given quantity) is independent of her own bids, embodying the first prescription of auction design.

dures (such as posted prices) may raise higher revenues. Nevertheless, in recent years, when economists and game theorists have been called upon to recommend selling procedures, most notably in cases of governments offering telecommunications spectrum, they have generally advocated using dynamic auctions. At the same time, with the rise in recent years of online bazaars such as eBay, casual empiricism suggests that dynamic auctions have gained market share at the expense of sealed-bid auctions, and that auction mechanisms have gained market share at the expense of non-auction mechanisms. Finally, the potential advantages of posted prices may be obtained in the auction procedure proposed in the current article by simply appending a reserve price or a supply curve to the otherwise efficient auction.

[2] As recounted by Philippe Jehiel and Benny Moldovanu (2000), ten licenses for virtually homogeneous spectrum were offered to the four German mobile phone incumbents. In the first round of bidding, Mannesmann placed high bids of DM 36,360,000 per MHz on each of licenses 1 through 5 and high bids of DM 40,000,000 per MHz on each of licenses 6 through 10. In the second round of bidding, T-Mobil raised Mannesmann on licenses 1 through 5 by bidding a price of DM 40,010,000 (the minimum bid increment was 10 percent), while letting Mannesmann maintain the high bids on licenses 6 through 10. In the third round of bidding, no new bids were entered, and so the auction ended in two rounds with the two largest incumbents dividing the market almost equally at an apparently uncompetitive low price.

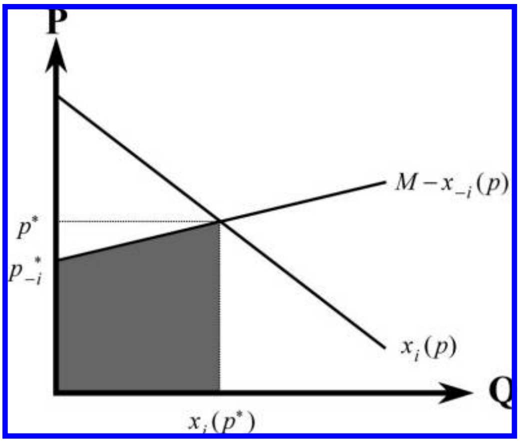

FIGURE 1. PAYMENT RULE IN THE VICKREY AUCTION

There would appear to be significant advantages, however, if a multi-unit auction format could also reflect the second prescription of auction design. The principal questions under consideration may thus be stated:

> *Can the analogy between the English auction and the second-price auction be completed for multiple units? In particular, when bidders have pure private values, does there exist a simple ascending-bid auction for homogeneous goods whose static representation is the (multi-unit) Vickrey auction? And, to the extent that the analogy can be completed, will the ascending-bid auction outperform the sealed-bid auction in interdependent values environments generalizing the Milgrom-Weber symmetric model?*

This article provides a substantial affirmative answer. A new ascending-bid auction is proposed, which operates as follows. The auctioneer calls a price, bidders respond with quantities, and the process iterates with increasing prices until demand is no greater than supply. A bidder's payment does *not*, however, equal her final quantity times the final price. Rather, at each price p, the auctioneer determines whether, for any bidder i, the aggregate demand $x_{-i}(p)$ of bidder i's rivals is less than the supply M. If so, the difference is deemed "clinched," and any goods newly clinched are awarded to bidder i at price p.

For example, suppose that two identical objects are available and that three bidders—A, B, and C—initially bid for quantities of 2, 1, and 1, respectively. Suppose that the bidders continue to bid these quantities until price p, when Bidder C reduces from 1 unit to 0, dropping out of the auction. While there continues to be excess demand, Bidder A's opponents now collectively demand only one unit, while two units are available. Bidder A therefore clinches one unit at price p, and the auction (for the remaining object) continues.

In the new auction design, a bidder's payment for inframarginal units is effectively decoupled from her bids for marginal units, eliminating any incentive for demand reduction. Consequently, with private values, sincere bidding by every bidder is an equilibrium, yielding the same efficient outcome as the Vickrey auction. Moreover, under incomplete information and a "full support" assumption, sincere bidding is the unique outcome of iterated weak dominance, just as sincere bidding is the unique outcome of weak dominance in the Vickrey auction. Thus, the new ascending-bid auction design has an analogous relationship to the (multi-unit) Vickrey auction that the English auction has to the second-price auction.

Furthermore, consider a symmetric setting in which bidders have constant marginal values that are *interdependent* in the sense that each bidder's value depends on her rivals' signals. (While restrictive, this model strictly generalizes the classic Milgrom and Weber framework.) Let M denote the supply of objects and let λ_i denote the number of objects desired by bidder i. If $\lambda_i \equiv \lambda$ and M/λ is an integer, efficient equilibria exist in both the static and dynamic auctions, but the seller's expected revenues are greater in the dynamic auction, replicating Milgrom and Weber's point. For the remaining parameter values, the new (dynamic) auction format outperforms the (static) Vickrey auction on efficiency: efficient equilibria exist in the dynamic auction, but are not present in the static auction.

Simplicity or transparency to bidders should be viewed as one important attribute and advantage of the proposed auction. While the single-item Vickrey auction is well known, the multi-unit version proposed by Vickrey in the same 1961 article remains relatively obscure even among economists, and is hardly ever advocated for real-world use. One reason seems to be that many believe it is too complicated for practitioners to understand, even in the private values environment where the traditional theory finds no informational advantages to a dynamic auction over a static auction.[3] By contrast, the ascending-bid design proposed here seems simple enough to be understood by any aficionado of baseball pennant races. This prediction appears to be borne out in the early experimental evidence (see Section VI).

Privacy preservation of the winning bidders' values is another attribute and advantage of the new ascending-bid auction. Noting that English auctions are quite prevalent in the real world while sealed-bid second-price auctions are comparatively rare, Michael H. Rothkopf et al. (1990) offer a possible explanation: bidders will be reluctant to reveal their private values truthfully in an auction if either there may be cheating by the auctioneer or there will be subsequent auctions or negotiations in which the information revealed can be used against them.[4] Such considerations favor ascending-bid auctions, since winning bidders need not reveal their entire demand curves, only the portion below the winning price.[5]

The following articles constitute a less-than-exhaustive list of related research. Milgrom and Weber (1982b, pp. 4–5) introduce the uniform-price ascending-bid auction when bidders have unit demands and there are multiple identical objects, and extend their (1982a) analysis of symmetric environments with affiliated information to this multi-unit context. Eric S. Maskin (1992) demonstrates that, even for single-item auctions with asymmetric bidders and interdependent information, the English auction is more likely to yield efficiency than the sealed-bid second-price auction. Maskin and John G. Riley (1989) examine optimal auctions for multiple identical objects in an independent private values setting. Alexander S. Kelso and Vincent P. Crawford (1982), Gabrielle Demange et al. (1986), Sushil Bikhchandani and John W. Mamer (1997), Bikhchandani (1999), Faruk Gul and Ennio Stacchetti (1999, 2000) and Milgrom (2000) study various auction procedures for multiple items and their relationship with Walrasian prices under complete information. Bikhchandani and Joseph M. Ostroy (2001) and Bikhchandani et al. (2002) formulate the auction problem as a linear programming problem and reinterpret the auction design herein as a primal-dual algorithm. Motty Perry and Philip J. Reny (2001, 2002) study more general homogeneous goods environments with interdependent values and extend the auction design herein to such environments. Partha Dasgupta and Maskin (2000) define a sealed-bid auction procedure designed to attain efficiency with heterogeneous items. Vijay Krishna and Perry (1998) study the Vickrey auction in an independent private values setting.

The article is organized as follows. Section I informally describes the new ascending-bid auction design via an illustrative example. Section II presents the formal model. Section III analyzes the private values case, demonstrating that sincere bidding is an equilibrium and that, under incomplete information and a "full support" assumption, it is the unique outcome of iterated weak dominance. Section IV treats, in a continuous-time formulation, a symmetric model

[3] Indeed, the subtlety of the Vickrey auction has been a problem even in experimental auction studies involving merely a single object. John H. Kagel et al. (1987) found that bidders with affiliated private values behaved closer to the dominant strategy in ascending-clock auctions than in sealed-bid second-price auctions.

[4] Richard Engelbrecht-Wiggans and Charles M. Kahn (1991) and Rothkopf and Ronald M. Harstad (1995) also provide models emphasizing the importance of protecting the privacy of winners' valuations.

[5] For example, suppose that the government sells a spectrum license valued by the highest bidder at $1 billion but by the second-highest bidder at only $100 million in a sealed-bid second-price auction. There are at least three potential problems here. First, there is likely to be a public relations disaster, as the ensuing newspaper headlines read, "Billion-dollar communications license given away for 10 cents on the dollar." Second, there may be a problem of seller cheating: after opening the submitted bids, the auctioneer may ask his friend, "Mind if I insert a bogus $997 million bid in your name? It won't cost you anything, but it will earn me a lot of money." Third, revelation of the winner's billion-dollar value may imperil her subsequent bargaining position with equipment suppliers. By contrast, an English auction avoids these problems, revealing only that the high bidder's value exceeded $100 million. (See also the nice discussion of this point in McMillan, 1994, especially p. 148.)

TABLE 1—BIDDER VALUATIONS IN ILLUSTRATIVE EXAMPLE

	Bidder A	Bidder B	Bidder C	Bidder D	Bidder E
Marginal value (1 unit)	123	75	125	85	45
Marginal value (2 units)	113	5	125	65	25
Marginal value (3 units)	103	3	49	7	5

of interdependent values, where bidders have affiliated signals and exhibit constant marginal values. Section V discusses the limitations of the interdependent-values analysis. Section VI concludes. Proofs appear in the Appendix.

I. An Illustrative Example

I will illustrate my proposal for an ascending-bid, multi-unit auction with an example loosely patterned after the first U.S. spectrum auction, the Nationwide Narrowband Auction. There are five identical licenses for auction.[6] Bidders have taste for more than one license, but each is limited to winning at most three licenses.[7] There are five bidders with values in the relevant range, and their marginal values are given as in Table 1 (where numbers are expressed in millions of dollars).

For example, if Bidder A were to purchase two licenses at a price of 75 each, her total utility from the transaction would be computed by: $u_A(1) + u_A(2) - 75 - 75 = 123 + 113 - 150 = 86$. In this example, bidders are presumed to possess complete information about their rivals' valuations.

The proposed auction is operated as an *ascending-clock auction.* The auctioneer announces a price, p, and each bidder i responds with a quantity, $q_i(p)$. The auctioneer then calculates the aggregate demand and increases the price until the market clears. Payments are calculated according to a "clinching" rule. Suppose that the auction begins with the auctioneer announcing a price of \$10 million ($+ \varepsilon$). Bidders A to E, if bidding sincerely according to the valuations of Table 1, would respond with demands of 3, 1, 3, 2, and 2, respectively. The aggregate demand of 11 exceeds the available supply of 5, so the auction must proceed further. Assume that the auctioneer increases the price continuously. Bidder E reduces his quantity demanded from 2 to 1 at \$25 million, Bidder E drops out of the auction completely at \$45 million, and Bidder C reduces his quantity demanded from 3 to 2 at \$49 million, yielding:

Price	Bidder A	Bidder B	Bidder C	Bidder D	Bidder E
49	3	1	2	2	0

The aggregate demand, now 8, continues to exceed the available supply of 5, so the price must rise further. When the price reaches \$65 million, Bidder D reduces her demand from 2 to 1, but the aggregate demand of 7 continues to exceed the available supply of 5:

Price	Bidder A	Bidder B	Bidder C	Bidder D	Bidder E
65	3	1	2	1	0

Let us examine this situation carefully, however, from Bidder A's perspective. The demands of all bidders *other than Bidder A* (i.e., $1 + 2 + 1 + 0$) total only 4, while 5 licenses are available for sale. If Bidders B to E bid monotonically, Bidder A is now mathematically guaranteed to win at least one license. In the language of this article (and in the standard language of American sports

[6] In actuality, the FCC's Nationwide Narrowband Auction offered ten licenses of three different types: five (essentially identical) 50/50 kHz paired licenses; three (essentially identical) 50/12.5 kHz paired licenses; and two (essentially identical) 50 kHz unpaired licenses. For an extraordinarily cogent discussion of the Nationwide Narrowband Auction, see Cramton (1995).

[7] In actuality, the FCC limited bidders to acquiring three licenses, either through the auction or through resale. Observe that the total number of licenses is not an integer multiple of each bidder's limitation on purchases, so with incomplete information, the inefficiency result of Ausubel and Cramton (2002, Theorem 1) is applicable, even if the marginal values for the first, second, and third licenses are equal.

writing), Bidder A has *clinched* winning one unit. The rules of the auction take this calculation quite literally, by awarding each bidder any units that she clinches, at the clinching price. Bidder A thus wins a license at $65 million.

Since there is still excess demand, price continues upward. With continued sincere bidding relative to the valuations in Table 1, the next change in demands occurs at a price of $75 million. Bidder B drops out of the auction, but the aggregate demand of 6 continues to exceed the available supply of 5:

Price	Bidder A	Bidder B	Bidder C	Bidder D	Bidder E
75	3	0	2	1	0

Again examine the situation from Bidder A's perspective. Her opponents collectively demand only 0 + 2 + 1 + 0 = 3 units, whereas 5 units are available. It may now be said that she has clinched winning 2 units: whatever happens now (provided that her rivals bid monotonically), she is certain to win at least 2 units. Hence, the auction awards a second unit to Bidder A at the new clinching price of $75 million. By the same token, let us examine this situation from Bidder C's perspective. Bidder C's opponents collectively demand only 3 + 0 + 1 + 0 = 4 units, whereas 5 units are available. He has clinched winning 1 unit: whatever happens now (provided that his rivals bid monotonically), he is certain to win at least 1 unit. Hence, the auction awards one unit to Bidder C at a price of $75 million.

There continues to be excess demand until the price reaches $85 million. Bidder D then drops out of the auction, yielding:

Price	Bidder A	Bidder B	Bidder C	Bidder D	Bidder E
85	3	0	2	0	0

At $85 million, the market clears. Bidder A, who had already clinched a first unit at $65 million and a second at $75 million, wins a third unit at $85 million. Bidder C, who had already clinched a first unit at $75 million, wins a second unit at $85 million. In summary, we have the following auction outcome:

	Bidder A	Bidder B	Bidder C	Bidder D	Bidder E
Units won	3	0	2	0	0
Payments	65+75 +85	0	75+85	0	0

Observe that the outcome is efficient: the auction has put the licenses in the hands of bidders who value them the most. Also observe that the new (dynamic) auction has exactly replicated the outcome of the (sealed-bid) Vickrey auction. Bidder A won her first unit at the third-highest rejected bid, her second unit at the second-highest rejected bid, and her third unit at the highest rejected bid. Bidder C won his first unit at the second-highest rejected bid and his second unit at the highest rejected bid.

One interesting observation is that if Bidder A had been subject to a binding *budget constraint* of strictly between $225 and $255 million in this example, then the standard Vickrey auction would have likely failed to deliver the efficient outcome. In the sealed-bid implementation asking bidders to report their valuations using downward-sloping demand curves, the budget-constrained Bidder A would have been unable to afford to report that her marginal value for a third unit exceeded $85 million and so the item would instead have been awarded to Bidder D. There is no such difficulty in the proposed ascending-bid auction: Bidder A would fail to win three units only if her budget constraint were less than $225 million, a limit so low that it would prevent her from paying the true opportunity cost of the third unit.[8]

[8] If a budget-constrained multi-unit bidder bids only against single-unit bidders without budget constraints, then the proposed ascending-bid auction yields increased efficiency and revenues as compared to the sealed-bid Vickrey auction. In general multi-unit environments with budget constraints, however, the effect is ambiguous for two reasons. First, the ascending-bid auction facilitates the expression of full valuations by multi-unit bidders. If single-unit bidders themselves face budget constraints, then the multi-unit bidder may already win more in the sealed-bid auction than is efficient—despite her own budget constraint—and the ascending-bid auction may then exacerbate this effect by relaxing her budget constraint.

Next, let us reexamine the example of Table 1, in order to see what outcome would have ensued if we had instead applied a uniform-pricing rule. As before, the auctioneer calls prices, bidders respond with quantities, and the price is incremented until p^* is reached, at which there is no excess demand. In a *uniform-price ascending-clock auction,* however, any bidder i assigned a final quantity x_i^* pays the amount $p^*x_i^*$. Then, there exists an equilibrium in which the auction concludes at a price of $75 million and with an inefficient allocation. More strikingly, if the example is perturbed so that $\Pr\{u_A(3) = k\} = \varepsilon$, for every $k = 76, \ldots, 84, 86$ and for small but positive ε, this inefficient equilibrium is the *unique* outcome of iterated weak dominance. By contrast, the same criterion—applied to the ascending auction with a clinching rule—selects the sincere bidding equilibrium (see Theorem 2 for the general argument).

To analyze the perturbed example, let us suppose that the prior bidding has been sincere and consider the game at a price of $75 million. The standing bids of Bidders A to E are 3, 1, 2, 1, and 0, respectively. Observe that it is weakly dominant for Bidder D to maintain a quantity of 1 for all prices less than $85 million and then to reduce her quantity to 0 (since, with the perturbed $u_A(3)$, Bidder A has a positive probability of reducing from 3 to 2 at any price between $75 and $85 million). Similarly, it is weakly dominant for Bidder B to reduce her quantity to 0 at $75 million. Following elimination of these strategies for Bidders D and B, Bidder A (if she has $u_A(3) = 85$) has two candidate optimal actions: she can continue to bid sincerely, winning 3 items at a price of $85 million (giving her surplus of $84 million); or she can reduce her demand, thereby immediately ending the auction and winning 2 items at a price of $75 million (giving her surplus of $86 million). Thus, uniform pricing uniquely gives a final price of $75 million and an inefficient allocation of goods of (2, 0, 2, 1, 0) as the outcome of iterated weak dominance.[9] By contrast, a clinching rule uniquely yields an efficient allocation (Theorem 2), and despite giving away one license at a bargain $65 million in this example, yields $10 million more in revenue.[10]

Most other conventional auction approaches also yield inefficient equilibria when applied to the example of Table 1. One approach is to sell the identical objects, one after another, by *successive* single-item English auctions. This, for example, is how Sotheby's attempted to auction seven satellite transponders in November 1981 (see Milgrom and Weber, 1982b). Observe that there is then a tendency toward *intertemporal* arbitrage, which lends the auction process a uniform-price character.

A more sophisticated approach is the *simultaneous multiple-round* (SMR) auction used by the Federal Communications Commission (FCC) to assign spectrum licenses.[11] Bidders successively name prices on individual items; and the bidding is not deemed to have concluded for *any* single item until it stops for *all* items. In such a format, there is an even stronger tendency toward arbitrage, so that similar items sell for similar prices. Most strikingly, in the real-world Nationwide Narrowband Auction on which Example 1 was patterned, the five virtually identical 50/50 kHz paired licenses each sold for exactly $80 million;[12] subsequent FCC auctions have displayed only minor amounts of price discrepancy for similar licenses.

To the extent that prices are arbitraged under either of these approaches, essentially the same inefficiencies should result as in the uniform-price ascending-bid auction. If either five successive single-item auctions or the FCC's SMR auction were used in Example 1, Bidder A does

Second, multi-unit bidders who bid against budget-constrained opponents may have incentive to overbid on some units in order to deplete their opponents' budgets. This complicated effect may occur in both the sealed-bid and ascending-bid auctions, rendering the comparison ambiguous. The effect of budget constraints on auction strategies and outcomes will be analyzed further in future work.

[9] In the full iterated weak dominance argument (omitted here, for brevity), we would argue also that neither Bidder A nor Bidder C reduces demand below 2 and that neither Bidder B nor Bidder D reduces demand below 1 before the price reaches $75 million.

[10] The revenue ranking of the alternative ascending-bid auction versus the uniform-price ascending-bid auction is ambiguous.

[11] See Cramton (1995) and Milgrom (2000).

[12] See Cramton (1995). Moreover, bidders could submit any integer prices in excess of a (nonbinding) minimum bid increment.

best by bidding only up to a price of \$75 million. With the five successive auctions, Bidder A might consequently lose the first three auctions to Bidder C (2 units) and Bidder D (1 unit); with the high marginal values out of the way, however, Bidder A assures herself of winning the last two auctions at \$75 million each. Similarly, with the SMR auction, two of Bidder A's \$75 million bids would not be outbid.

Finally, let us consider the two sealed-bid auction formats that have generally been used for U.S. (and other governments') Treasury auctions: the *pay-as-bid auction* and the *uniform-price auction.*[13] As applied to Example 1, these two auctions again have the property that all equilibria in undominated strategies are inefficient. For the uniform-price auction, this follows the same argument as before: in an efficient equilibrium in undominated strategies, Bidder D's bid of \$85 million and Bidder B's bid of \$75 million need to be rejected. Bidder A calculates that she can improve her demand so that she wins only two items. For the pay-as-bid auction, we can apply almost identical reasoning: in an efficient equilibrium, the winning bids must all be at least \$85 million; otherwise, unsatisfied Bidder D could profitably deviate. But Bidder A could then substitute two bids of \$75 million ($+\ \varepsilon$) for her three bids of \$85 million, improving her payoff.[14]

II. The Model

There are at least two useful ways to formulate a mathematical model of the new auction. In the first, the price clock advances in discrete (e.g., integer) steps, and bidders' marginal valuations are taken from the same set of discrete (e.g., integer) values. In the second, the price clock operates in continuous time, enabling full separation of bidders' continuous signals. Sections II and III focus on the first, simpler formulation. The model is sufficiently rich to provide a relatively comprehensive treatment of private values. Moreover, most practical implementations of auctions include some amount of discreteness, giving rise to a positive probability of ties, and the analysis of the discrete formulation includes a rather complete treatment of ties. (Continuous-time games obviate this problem, since a tie is then a zero-probability event.) Section IV introduces and studies the second formulation, which is required for a treatment of efficiency using the standard (continuous) models of interdependent values.

A seller wishes to allocate M homogeneous goods among n bidders, $N \equiv \{1, \ldots, n\}$. Each bidder i may be assigned any quantity x_i in the consumption set X_i, subject to the feasibility constraint that $\sum_{i=1}^{n} x_i \leq M$. We simultaneously treat two interesting cases: $X_i = [0, \lambda_i]$, so that the good is perfectly divisible; or $X_i = \{0, 1, 2, \ldots, \lambda_i\}$, so that the good is discrete. (In either case, $0 < \lambda_i \leq M$.) Bidder i's utility is assumed to be quasilinear, equaling her pure private value, $U_i(x_i)$, for the quantity x_i she receives less the total payment, y_i, that she is obligated to pay: $U_i(x_i) - y_i$. The value $U_i(\cdot)$ is assumed to be the integral of a marginal value $u_i(\cdot)$, and so $U_i(x_i) = \int_0^{x_i} u_i(q)\, dq$. Each bidder's marginal value function, $u_i(\cdot)$, may be publicly known, making this a game of complete information, or privately known, making this a game of incomplete information. In either case, we assume that all marginal values are uniformly bounded above by $\bar{u} > 0$ and below by zero. The marginal value $u_i : [0, \lambda_i] \rightarrow \mathbb{Z}$ is assumed to be *weakly decreasing* in q and *integer valued.* Thus, bidders exhibit diminishing marginal utilities, Walrasian price(s) are guaranteed to exist, and the lowest Walrasian price is an integer between 0 and $\bar{u}$.

In order to simplify the following presentation, we will place two constraints on bidding strategies. First, bidding will be constrained by a monotone activity rule—equation (1) below—that is, bidders will be required to bid (weakly) downward-sloping demand curves. Second, bidders will be constrained not to bid for smaller quantities than they have already clinched (equation [5] below). While neither constraint is required for the results established

[13] These formats are defined and studied in detail in Ausubel and Cramton (2002).

[14] The pay-as-bid and uniform-price auctions also possess efficient equilibria, but only if we allow bidders to use weakly dominated strategies. For Example 1, following Bikhchandani (1999), it is an efficient equilibrium of the pay-as-bid auction if Bidder A submits three (winning) bids of 85 ($+\ \varepsilon$), Bidder C submits two (winning) bids of 85 ($+\ \varepsilon$), Bidder D submits a (losing) bid of 85, and some bidder submits an additional (losing) bid, b, of at least 76. Observe, however, that this requires that the additional losing bid, b, exceed the bidder's marginal value. If $b < 76$, Bidder A can profitably deviate by instead bidding only $b + \varepsilon$, thereby settling for only two objects. Similar reasoning applies to the uniform-price auction.

in this section,[15] both constraints simplify the description of the auction to the economist or to the bidder, and both would likely be imposed in real-world auctions.[16]

The auction is modeled as a dynamic game in discrete time. At each time $t = 0, 1, 2, \ldots, T$, the price $p^t = t$ is communicated to (or already known by) the n bidders, and each bidder i responds by bidding a quantity $x_i^t \in X_i$. The presentation is simplest if bidders are constrained to bid monotonically:

(1) *Monotone activity rule*: $x_i^t \leq x_i^{t-1}$,

for all $i = 1, \ldots, n$

and all $t = 1, \ldots, T$.

The final time, T, after which the auction exogenously ends, is selected so as not to bind, i.e., $T > \bar{u}$.

Suppose that the auction is fully subscribed at the starting price: $\sum_{i=1}^n x_i^0 \geq M$. Then the auction continues until such time that there is no excess demand, or until the exogenous ending time T is reached, whichever occurs sooner. Thus, after each time $t = 0, \ldots, T$, the auctioneer determines whether $\sum_{i=1}^n x_i^t \leq M$. If this inequality is satisfied, then the current time t is designated the *last* time of the auction: we write $L = t$. Each bidder i is assigned a final quantity x_i^* that satisfies $x_i^L \leq x_i^* \leq x_i^{L-1}$ and $\sum_{i=1}^n x_i^* = M$.[17] If this inequality is not satisfied but $t = T$, the exogenous ending time, then the current time T is designated the *last* time of the auction: we write $L = T$. Each bidder i is assigned a final quantity x_i^* that satisfies $x_i^* \leq x_i^L$ and $\sum_{i=1}^n x_i^* = M$.[18] Finally, if there remains excess demand and $t < T$, the auction game proceeds to time $t + 1$ (with associated price $p^{t+1} = t + 1$) and the process repeats.

At any time t, we define the vector of *cumulative clinches*, $\{C_i^t\}_{i=1}^n$, by:

$$(2) \qquad C_i^t = \max\left\{0, M - \sum_{j \neq i} x_j^t\right\},$$

for all $t = 0, \ldots, L - 1$ and i=1, ... , n, and

$$(3) \quad C_i^L = x_i^*,$$

where L is the last auction round and

x_i^* is the final quantity assigned to bidder i.

We define the vector of *current clinches*, $\{c_i^t\}_{i=1}^n$, at time t as the difference between the cumulative clinches at time t and the cumulative clinches at time $t - 1$, i.e.,

$$(4) \qquad c_i^t = C_i^t - C_i^{t-1},$$

for $t = 1, \ldots, L$ and $c_i^0 = C_i^0$, for all $i = 1, \ldots, n$.

As discussed above, in order to simplify the presentation, the bidder is also constrained to bid no smaller a quantity than her prior cumulative clinches:

$$(5) \quad x_i^t \geq C_i^{t-1},$$

for all $i = 1, \ldots, n$ and all $t = 1, \ldots, T$.

[15] The constraint of a monotone activity rule is dropped in the sequel paper, Ausubel (2002), where clinching is replaced by notions of "crediting" and "debiting," yet similar efficiency results are obtained. The constraint of not allowing bidding for smaller quantities than have already been clinched is irrelevant when suitable restrictions are placed on the rationing rule, for example, as in footnotes 17 and 18.

[16] In real-world multi-item auctions, activity rules are often imposed. The concern is that without an activity rule, a bidder with serious interest in the items for auction may choose to wait to bid, as a "snake in the grass," until the auction appears nearly ready to close. The activity rule prevents a bidder from concealing her true intentions until late in the auction, by requiring her to bid on a given quantity early in the auction in order to preserve the right to bid on this quantity late in the auction.

[17] If $\sum_{i=1}^n x_i^L < M$, then there is a need for rationing some of the bidders in order to sell the entire quantity M. So long as it is consistent with $x_i^L \leq x_i^* \leq x_i^{L-1}$ and $\sum_{i=1}^n x_i^* = M$, the rationing rule may be specified relatively arbitrarily, but it must satisfy the following monotonicity property: if $x_i^L < x_i^{L-1}$, then the expected quantity $E[x_i^*]$ assigned to bidder i must be strictly greater than her final bid x_i^L, and if the final bid x_i^L of bidder i is increased, while holding the final bids x_{-i}^L of all opposing bidders fixed, then the (probability distribution on the) quantity x_i^* assigned to bidder i must increase.

[18] The rationing rule may be specified relatively arbitrarily, but it must satisfy the following monotonicity property: if the final bid x_i^L of bidder i is increased, while holding the final bids x_{-i}^L of all opposing bidders fixed, then the (probability distribution on the) quantity x_i^* assigned to bidder i must increase.

The payment rule is that the payment for each unit is the price at which it is clinched. In general (including continuous time) games, let $p(t)$ denote the price at time t and let $C_i(t)$ denote the cumulative clinches based on the bids at time t. Then bidder i's total payment is the following Stieltjes integral:

$$y_i = \int_0^L p(t)\, dC_i(t). \tag{6}$$

In the discrete-time notation of the current section, the payment equation (6) may equivalently be written:

$$y_i = \sum_{t=0}^{L} p^t c_i^t. \tag{7}$$

However, if $\sum_{i=1}^n x_i^0 < M$, then each bidder i is assigned the quantity x_i^0 at the starting (zero) price.

A full specification of an ascending-bid auction game also requires some precision in stipulating the informational assumptions, as different informational assumptions potentially lead to different outcomes. Let $h^t \equiv \{x_1^{t'}, \ldots, x_n^{t'}\}_{t'<t}$ denote the history of play prior to time t and let h_i^t denote the summary of the history that is made observable to bidder i (*above and beyond her own prior bids*). The following are three of the most interesting available informational rules:

> FULL BID INFORMATION: The summary of the history observable to bidder i is: $h_i^t = \{x_1^{t'}, \ldots, x_n^{t'}\}_{t'<t}$, i.e., the complete history of all bids by all bidders.
>
> AGGREGATE BID INFORMATION: The summary of the history observable to bidder i is: $h_i^t = \{\sum_{j=1}^n x_j^{t'}\}_{t'<t}$, i.e., the complete history of the aggregate demand of all bidders.
>
> NO BID INFORMATION: The summary of the history observable to bidder i is: $h_i^t = 1$, if $\sum_{j=1}^n x_j^{t-1} > M$, and 0, otherwise, i.e., whether the auction is still open.

Given the informational assumption chosen, let H_i^t denote the set of all possible histories observable to bidder i at time t. In each of the theorems below, if the informational assumption is not otherwise specified, then the result holds for all three informational rules.

A *strategy* $\sigma_i : \{0, \ldots, T\} \times H_i^t \rightarrow X_i$ of player i ($i = 1, \ldots, n$) is any function of times and observable histories to quantities that is consistent with the bidding constraints, and the *strategy space* Σ_i is the set of all such functions $\sigma_i(t, h_i^t)$. The information structure of the auction game may be one of complete or incomplete information regarding opposing bidders' valuations. With complete information, each bidder is fully informed of the functions $\{U_j(\cdot)\}_{j=1}^n$, and (if there is also full bid information) the appropriate equilibrium concept is subgame perfect equilibrium. With incomplete information and pure private values, each bidder i is informed only of her own valuation function $U_i(\cdot)$ and of the joint probability distribution $F(\cdot)$ from which the profile $\{U_j(\cdot)\}_{j=1}^n$ is drawn. In static games of incomplete information, authors sometimes advocate *ex post equilibrium,* which requires that the strategy for each player would remain optimal if the player were to learn her opponents' types (see Jacques Crémer and Richard P. McLean, 1985). In the current context of a dynamic game, the equilibrium concept that we will define and use is *ex post perfect equilibrium,* which imposes this same condition at every node of the auction game:

EX POST PERFECT EQUILIBRIUM. The strategy n-tuple $\{\sigma_i\}_{i=1}^n$ is said to comprise an ex post perfect equilibrium if for every time t, following any history h^t, and for every realization $\{U_i\}_{i=1}^n$ of private information, the n-tuple of continuation strategies $\{\sigma_i(\,\cdot\,, \cdot\,|t, h_i^t, U_i)\}_{i=1}^n$ constitutes a Nash equilibrium of the game in which the realization of $\{U_i\}_{i=1}^n$ is common knowledge.

Alternatively, we could have explicitly defined beliefs for each bidder and stated the theorems of this article in terms of the perfect Bayesian equilibrium concept.[19] (Indeed, the results in

[19] We would begin by specifying that, after every history, each player i has posterior *beliefs,* denoted $\mu_i(\,\cdot\,|t, h_i^t, U_i)$, over opponents' utility functions, $U_{-i}(\cdot) \equiv \{U_j(\cdot)\}_{j\neq i}$. The n-tuple $\{\sigma_i, \mu_i\}_{i=1}^n$ is then defined to comprise a *perfect Bayesian equilibrium* if the strategies $\sigma_i \in \Sigma_i$, if the beliefs μ_i are updated by Bayes' rule whenever possible, and if following any history h^t of play prior to time t, σ_i is a best

Section IV involving interdependent valuations will utilize perfect Bayesian equilibrium.) Stating the private values results in their current form, however, gives them a number of additional desirable properties, e.g., the results are independent of the underlying distributions of bidders' types (see also Crémer and McLean, 1985; Maskin, 1992; Ausubel, 1999; Perry and Reny, 2001, 2002). The results as stated also encompass the complete-information version of the model, since ex post perfect equilibrium then reduces to the familiar equilibrium concept of subgame perfect equilibrium.

III. Results with Private Values

This section provides the private values results of the article. Sincere bidding is an ex post perfect equilibrium of the model of Section II. Furthermore, under incomplete information and a "full-support" assumption, it is the unique outcome of iterated weak dominance. We begin by defining sincere bidding, which informally means "you just bid what you think it is worth":

DEFINITION 1: The *sincere demand* of bidder i at price p is: $Q_i(p) \equiv \inf\{arg\ max_{x_i \in X_i} \{U_i(x_i) - px_i\}\}$. *Sincere bidding* is the strategy in which, subject to the constraints posed by the monotone activity rule and her previous clinches, bidder i bids her sincere demand at every time t and after every history h_i^t:

$$x_i^t = \min\{x_i^{t-1}, \max\{Q_i(p^t), C_i^{t-1}\}\}, \quad (8)$$

$$\text{for all } t = 1, \ldots, T, \text{ and } x_i^0 = Q_i(p^0).$$

For example, given the illustrative valuations of Bidder D in Table 1, the sincere bid is: $x_D^t = 3$, for $t = 0, \ldots, 6$; $x_D^t = 2$, for $t = 7, \ldots, 64$; $x_D^t = 1$, for $t = 65, \ldots, 84$; and $x_D^t = 0$, for $t = 85, \ldots, T$. This, however, assumes that the constraint $C_D^{t-1} \leq x_D^t \leq x_D^{t-1}$ is not binding due to the history of play in the auction. Sincere bidding is specified in Definition 1 so that the bidder never bids more than her quantity in the previous period and never bids less than the quantity that she has already clinched.

response (given beliefs) for every player i in the continuation game against $\{\sigma_j\}_{j \neq i}$.

Note that the assignment of goods in the auction has been specified (see Section II, para. 5) in such a way that bidder i may be required to purchase more than $x_i^* > x_i^L$ units at price p^L (that is, a larger number of units than she bid for at that price, albeit no larger a number than x_i^{L-1}, the number she bid for at the previous price). Nevertheless, observe that, given the sincere-bidding strategy specified in Definition 1, there is never any ex post regret. Any time t in which a sincere bidder i reduces her bid, she is indifferent between receiving her prior bid x_i^{t-1} and her current bid x_i^t. For example, using the strategy in the previous paragraph, Bidder D could potentially win 2 units at a price of 65, even though $x_D^{65} = 1$. Her marginal value equals 65, however, so she is in fact indifferent to winning 1 or 2 units at 65.

We are now ready to state our first theorem. All of the theorems are proved in the Appendix.

THEOREM 1: In the alternative ascending-bid auction with private values, sincere bidding by all bidders is an ex post perfect equilibrium,[20] yielding the efficient outcome of the Vickrey auction. Furthermore, with no bid information, sincere bidding is a weakly dominant strategy for every bidder after every history.

Theorem 1 notwithstanding, there may exist other equilibria besides the sincere-bidding equilibrium. Consider the following example with two bidders, subscripted by A and B, and two identical items. Suppose that $u_A(1) = 4$; $u_A(2) = 2$; $u_B(1) = 3$; and $u_B(2) = 1$. There is a sincere-bidding equilibrium in which bidder A bids for two units at $t = 0, 1$; one unit at $t = 2, 3$; and zero units at $t \geq 4$. Bidder B bids for two units at $t = 0$; one unit at $t = 1, 2$; and zero units at $t \geq 3$. Each bidder wins one unit, with Bidder A paying 1 and Bidder B paying 2. However, there also exists a "low revenue" equilibrium in which Bidder A bids for one unit at $t = 0, 1, 2, 3$, and zero units at $t \geq 4$, while Bidder B bids for one unit at $t = 0, 1, 2$, and zero units at $t \geq 3$. In the low-revenue equilibrium, each bidder again wins one unit, but each bidder pays zero. There also exists a continuum of other equilibria

[20] With full bid information and under complete information regarding bidders' valuations, this statement simplifies to saying that sincere bidding by every bidder is a subgame perfect equilibrium.

in this example.[21] Each of these equilibria corresponds to an equilibrium of the Vickrey auction. For example, the low-revenue equilibrium corresponds to the equilibrium of the Vickrey auction in which Bidder A submits bids of 4 and 0, and in which Bidder B submits bids of 3 and 0.

In the Vickrey auction, the additional equilibria are discarded by eliminating (weakly) dominated strategies.[22] More intricate reasoning is generally required to eliminate the additional equilibria in the alternative ascending-bid auction. The reason is that, with full or aggregate bid information, insincere bidding is not necessarily dominated. For example, suppose that for some bizarre reason, bidder j uses the strategy of maintaining $x_j^t = x_j^0$ so long as $x_i > K$, for some positive constant K, but of dropping to $x_j^{t+1} = 0$ in the first period following that $x_i^t \leq K$. Then it is possible that bidder i may improve her payoff by reducing her demand to K at a price p where her marginal utility, $u_i(K)$, still exceeds p.

The additional equilibria may be eliminated, however, using a combination of *iterated* weak dominance and *incomplete* information. In the above example, suppose that the marginal values of the respective bidders are instead distributed according to:

$$(u_A(1),\ u_A(2)) = \begin{cases} (4, 2), \text{ with probability } 1 - 11\varepsilon \\ (4, 1), \text{ with probability } \varepsilon \\ (4, 0), \text{ with probability } \varepsilon \\ (3, 2), \text{ with probability } \varepsilon \\ (3, 1), \text{ with probability } \varepsilon \\ (3, 0), \text{ with probability } \varepsilon \\ (2, 2), \text{ with probability } \varepsilon \\ (2, 1), \text{ with probability } \varepsilon \\ (2, 0), \text{ with probability } \varepsilon \\ (1, 1), \text{ with probability } \varepsilon \\ (1, 0), \text{ with probability } \varepsilon \\ (0, 0), \text{ with probability } \varepsilon \end{cases},$$

$$(u_B(1),\ u_B(2)) = \begin{cases} (3, 1), \text{ with probability } 1 - 6\varepsilon \\ (3, 0), \text{ with probability } \varepsilon \\ (2, 1), \text{ with probability } \varepsilon \\ (2, 0), \text{ with probability } \varepsilon \\ (1, 1), \text{ with probability } \varepsilon \\ (1, 0), \text{ with probability } \varepsilon \\ (0, 0), \text{ with probability } \varepsilon \end{cases}.$$

Then it is straightforward to see that, with a single round of elimination of weakly dominated strategies, one can eliminate the possibility that either bidder will prematurely reduce demand from one unit to zero. With a second round of elimination, one can eliminate the possibility that either bidder will prematurely reduce demand from two units to one—once her opponent has already reduced to one unit. And with a third round of elimination, one can eliminate the possibility that either bidder will prematurely reduce demand from two units to one—before her opponent has already reduced to one unit.

More generally, let us assume incomplete information and make the following assumption:

DEFINITION 2: For any nonnegative integer k, let $\Phi(k)$ denote the set of all weakly decreasing functions $\varphi : X_i \rightarrow \{0, \ldots, k\}$. In the private values model with incomplete information, bidder j is said to satisfy the *full support assumption* if there exists $\bar{u}_j \geq 0$ such that the probability distribution from which bidder j's marginal value function $u_j(\cdot)$ is drawn has support equal to the full set $\Phi(\bar{u}_j)$.[23]

The role of the full support assumption is to guarantee that, conditional on a sincere bid, $x_j(t)$, at time t, there is both a positive probability that the next sincere bid satisfies $x_j(t+1) > x_j(t) - \varepsilon$ (provided, of course, that $t + 1 < \bar{u}_j$) and a positive probability that the next sincere bid satisfies $x_j(t+1) < \varepsilon$, for every $\varepsilon > 0$. If the full support assumption holds for all bidders $j \neq i$, then every bid by bidder i matters.[24] The next

[21] In addition, there exists a third (pure) equilibrium strategy for bidder A, in which bidder A bids for two units at $t = 0$; one unit at $t = 1, 2, 3$; and zero units at $t \geq 4$. A continuum of equilibria is constructed by pairing each mixture for bidder A over the three aforementioned pure strategies with each mixture for bidder B over the two aforementioned pure strategies. I am grateful to an anonymous referee for providing this example.

[22] For example, bidder A submitting bids of 4 and 0 is weakly dominated by bidder A (sincerely) submitting bids of 4 and 2, since if bidder B (unexpectedly) submitted bids of 1 and 1, bidder A would then attain a higher payoff.

[23] If, instead of being drawn independently, the utility functions of the bidders are drawn from a joint probability distribution, then the analogous condition can be required on the marginal distribution for bidder i, given any realization for bidders $-i$.

[24] Conversely, suppose in an example similar to that of Table 1 that Bidder D's marginal value for a second unit

theorem shows that, under private values, incomplete information and the full support assumption, sincere bidding is the unique outcome of iterated weak dominance:

THEOREM 2: Under private values, incomplete information and the full support assumption, sincere bidding by all bidders is the unique outcome of iterated elimination of weakly dominated strategies in the alternative ascending-bid auction.[25]

Observe the following special cases of Theorem 2. For a single item, sincere bidding is the unique outcome of iterated weak dominance in the English auction. For M identical items and bidders with unit demands, sincere bidding is the unique outcome of iterated weak dominance in the uniform-price ascending auction. Both of these special cases obviously also require the full support assumption, which specialized to these cases is simply the requirement that the support of $u_i(1)$ is a set $\{0, \ldots, \bar{u}_i\}$.

One other observation is worth making at this juncture. The reason that we are able to obtain exactly the Vickrey outcome (as opposed to merely an approximation) in the discrete auction game is our assumption that all marginal valuations are integers. As a consequence, all payments in the Vickrey auction are integers, and there is no loss of information in eliciting bids only at integer prices. In the interdependent values model of the next section, however, marginal valuations may take any nonnegative real values, so it is then necessary to utilize a continuous game in order to obtain full efficiency.

IV. Results in a Continuous-Time Game with Symmetric, Interdependent Values

Among the most influential results in the single-item auction literature is the comparison between the second-price auction (a static auction) and the English auction (the associated dynamic auction) in a symmetric model with affiliated values. Each format exhibits an efficient equilibrium, but the efficient equilibrium of the dynamic auction yields higher expected revenues than that of the static auction (Milgrom and Weber, 1982a).[26] The analogous comparison, for auctions of multiple identical objects, is the comparison between the Vickrey auction and the alternative ascending-bid auction of this article. We find in this section that, in a symmetric model with flat demands and affiliated values, the dynamic auction has two advantages over the static auction. First, exactly as in Milgrom and Weber's analysis, the dynamic auction provides greater linkage between the payment and the bidders' signals, increasing the seller's expected revenues. Second, a "Champion's Plague" (or generalized Winner's Curse) emerges that is not present in the single-item analysis, adversely affecting the efficiency of the static auction.

A seller offers M discrete (and indivisible) units of a homogeneous good. The n bidders have "flat demands": each bidder i obtains constant marginal utility of V_i from each of up to λ_i units of the good, but zero marginal utility from any more than λ_i units, where the capacity λ_i satisfies $0 < \lambda_i \leq M$. Let the capacities be sufficiently large that there is competition for every unit of the good (i.e., $\sum_{j \neq i} \lambda_j \geq M$). The marginal values V_i $(i = 1, \ldots, n)$ are assumed to derive from affiliated signals. Let $\mathbf{S} \equiv (S_1, \ldots, S_n)$ be a vector of n real-valued signals which are privately observed by the n respective bidders. Also let $\mathbf{S_{-i}}$ denote the $(n - 1)$ signals other than that observed by bidder i, without the identities of the individual bidders indicated. Following Milgrom and Weber (1982a), it will be assumed that:

(A.1) $V_i = v(S_i, \mathbf{S_{-i}})$, where $v(\cdot)$ is the same nonnegative-valued function for every bidder i $(i = 1, \ldots, n)$, $v(\cdot)$ is continuous in all its arguments, $v(\cdot)$ is strictly in-

equals 65 but that there is zero probability that any opposing bidder's marginal value is anywhere in the interval [60, 70]. Then sincere bidding is not quite mandated: it is irrelevant to Bidder D's payoff at which price in [60, 70] she reduces her quantity from 2 to 1.

[25] Elimination or iterated elimination of *weakly* dominated strategies is sometimes criticized because its outcome may depend on the order of elimination. Observe, however, that Theorem 2 establishes order independence in this particular context.

[26] Milgrom and Weber (1982a) also compare the sealed-bid first-price auction of a single item. For a comparison involving the pay-as-bid auction (the multi-unit version of the first-price auction), see Ausubel and Cramton (2002).

creasing in its first argument, and $v(\cdot)$ is nondecreasing in its remaining arguments.[27]

(A.2) $E[V_i] < \infty$, for every bidder i ($i = 1, \ldots, n$).

(A.3) The variables $(S_1, \ldots, S_n)$ are affiliated.

(A.4) The joint density, $f(\cdot, \ldots, \cdot)$, of $(S_1, \ldots, S_n)$ is symmetric in its arguments.

Loosely speaking, the affiliation assumption (A.3) requires that the agents' signals, S_i, be nonnegatively correlated with one another. More precisely, let s and s' be possible realizations of $(S_1, \ldots, S_n)$. Let $s \vee s'$ and $s \wedge s'$ denote their componentwise maximum and minimum, respectively. We say that $(S_1, \ldots, S_n)$ are *affiliated* if $f(s \vee s')f(s \wedge s') \geq f(s)f(s')$, for all s and s' (see Milgrom and Weber, 1982a, p. 1098).

It will often be helpful to let $\mathbf{S}_{-\mathbf{ij}}$ denote the $(n - 2)$-tuple of signals received by bidders other than i and j, and to assume also:

(A.5) $V_i = v(S_i, S_j, \mathbf{S}_{-\mathbf{ij}}) > v(S_j, S_i, \mathbf{S}_{-\mathbf{ij}}) = V_j$, whenever $S_i > S_j$.

In order to accommodate full separation by bidders in their signals—which is necessary for efficiency in a model with a continuum of signals—we change our model to a continuous-time game. But, once going to a continuous-time game, we need to treat the possibility that bidder i's strategy may be to reduce her quantity at a given time, while bidder j's strategy may be to reduce his quantity at the soonest possible instant after bidder i reduces her quantity. Ideally, we should allow "moves that occur consecutively but at the same moment in time" (Leo K. Simon and Maxwell B. Stinchcombe, 1989, p. 1181).

The game may thus be conceptualized by thinking of "time" as being represented by a pair, (t, r), where t is given by a continuous ascending clock and r is given by an implicit discrete ascending counter. Times are ordered lexicographically: first in t; and second in r. Generally speaking, the clock time t increments continuously, and each bidder is free to reduce her quantity at any clock time. If, however, bidder i reduces her quantity at a given clock time t, bidder j is allowed to respond by reducing his quantity at the same clock time t (but, nevertheless, after bidder i's move). This is the role of the counter r: if bidder i reduces her quantity at (t, r), the next available time that follows is $(t, r + 1)$. Each time that some bidder reduces her quantity, the counter increments instead of the clock; and when players have finished reducing their quantities at the current clock time, the clock resumes instead.

Whenever the clock is ascending, we shall make full use of the continuous-time framework by specifying that price is a continuous and strictly-increasing function of time and, in fact, for simplicity $dp/dt = 1$. There is no conceptual or game-theoretic reason, however, why the clock needs to resume at the same price as where it stopped and, in fact, we have considerable latitude in specifying the price at which the clock resumes. We shall consider two variations on the continuous-time auction rules:

> STOPPING THE CLOCK: Whenever a bidder reduces her quantity demanded, the price clock pauses to enable other quantity reductions. The clock then resumes its ascent at the same price where it stopped, so that $p(t) = t$.
>
> TURNING BACK THE CLOCK: The same procedure is followed as in "stopping the clock." Whenever any bidder reduces her quantity to *zero,* however, the price clock is restarted at *zero,* so that $p(t) = t - t_0$, where t_0 is the most recent time at which some bidder has reduced to zero.[28]

There will be no confusion if we suppress the implicit counter r from our notation. Any history of the auction game can be uniquely summarized by a finite string of pairs: $h \equiv (t^0; x^0), \ldots, (t^L; x^L)$. Each t^ℓ denotes the time of the ℓ^{th} occasion on which one or more bidders strictly decreased her quantity, and each x^ℓ de-

[27] As noted by Milgrom and Weber (1982a, p. 1100), the "nondegeneracy assumption" that a bidder's expected value is strictly increasing in her own signal is unnecessary for the results to hold, but greatly simplifies the proofs.

[28] In the working paper version (Ausubel, 1997), arbitrary restarting prices were allowed as a function of the history, in the "turning back the clock" variation. Here, we restrict attention to restarting prices of zero, in order both to simplify the exposition and to limit the information about the bidders' distributions that needs to be known by the auctioneer.

notes the vector of quantities demanded by bidders 1, ... , n beginning at that occasion. Since quantities are discrete, and given the monotone activity rule, the length of any history h is bounded by the number of objects. Together with the current time, h fully summarizes the prior play of the game.[29] Further, let $x_i(h)$ denote bidder i's current quantity and let $C_i(h) = M - \sum_{j \neq i} x_j(h)$ denote bidder i's cumulative clinches after history h.

Let s_i denote the realization of the private signal S_i received by bidder i, and let h_i denote the summary of the history which is made observable to bidder i. A *pure strategy* for bidder i is a pair of functions, $\beta_i(s_i, h_i)$ and $\gamma_i(s_i, h_i)$, of private signals and observable summaries of the history. Then $\beta_i(s_i, h_i)$ provides the earliest price at which bidder i will decrease her demand, and $\gamma_i(s_i, h_i)$ provides the quantity to which she will decrease her demand, under the hypothesis that no opposing bidder j reduces his own quantity first. (It need not indicate what bidder i will do if some opposing bidder does reduce his quantity first, since then the history changes from h to h', and so bidder i would instead play according to $(\beta_i(s_i, h_i'), \gamma_i(s_i, h_i'))$. The function $\gamma_i(s_i, h_i)$ is restricted to generate bids that are consistent with the monotone activity rule and that are never bids for fewer units than have already been clinched. Payments are defined analogously as in Section II.

For bidders with private values (that is, $V_i = v(S_i)$), Theorems 1 and 2 continue to hold in the continuous-time framework of this section. Theorem 2 is established in this framework by first eliminating all strategies except sincere bidding following histories such that the aggregate demand of all bidders equals $M + 1$, then eliminating all strategies except sincere bidding following histories such that the aggregate demand of all bidders equals $M + 2$, etc. The proofs are omitted from the article.

For bidders with interdependent values (that is, $V_i = v(S_i, \mathbf{S_{-i}})$ nontrivially depends on $\mathbf{S_{-i}}$), the analysis dichotomizes into two cases:[30] (i) bidders have identical capacities ($\lambda_i = \lambda$) and the number of available items is an integer multiple of the bidders' capacities; and (ii) bidders' capacities are unequal or the number of available items is not an integer multiple of the bidders' capacities. The intuitive explanation for the difference between these two cases is that when $m \equiv M/\lambda$ is an integer, the model is closely related to one in which bidders possess *unit* demands and compete for m indivisible objects. There, symmetric equilibria with flat bid functions exist, yielding full efficiency. When $m \equiv M/\lambda$ is *not* an integer or the λ_i are *not* equal, however, symmetric equilibria with flat bid functions no longer exist.

First, in the case where M/λ is an integer, the static and dynamic auctions both have efficient equilibria but, due to the affiliated values, the expected revenues of the dynamic auction are greater:

THEOREM 3: Let the bidders have identical capacities, λ, and let $m \equiv M/\lambda$ be an integer. Then in a symmetric model with interdependent values satisfying (A.1)–(A.4), both the Vickrey auction and the alternative ascending-bid auction have efficient equilibria. Each attains full efficiency, but the alternative ascending-bid auction raises expected revenues the same as or higher than the Vickrey auction.[31]

In the case where M/λ is not an integer, or the λ_i are not equal, however, the static auction does not admit efficient equilibria in the interdependent values models, while the dynamic auction possesses an efficient equilibrium. The intuition for the result builds upon the celebrated "Winner's Curse":

[29] To see the richness of the histories that this framework and notation allow, consider the history $h = (0; 3, 2, 2, 2)$, $(15; 3, 1, 2, 2)$, $(15; 3, 1, 2, 0)$, $(27; 3, 0, 2, 0)$. This should be interpreted as saying that Bidders A to D began by bidding 3, 2, 2, and 2, respectively. At a price of 15, Bidder B reduced from 2 units to 1. Immediately after—and at the same price of 15—Bidder D responded by reducing from 2 units to 0. The interpretation of the remaining point in history h depends on whether we are using the "stopping-the-clock" or the "turning-back-the-clock" variation. With stopping the clock, $p(27) = 27$, meaning that Bidder B reduced from 1 unit to 0 at a price of 27. With turning back the clock, $p(27) = 27 - 15 = 12$ (since $t = 15$ is the most recent time at which some bidder has reduced to zero), meaning that Bidder B reduced from 1 unit to 0 at a price of 12.

[30] This dichotomy parallels our analysis of the uniform-price auction in Ausubel and Cramton (2002).

[31] Theorem 3 holds regardless of whether the rule governing the price process is "stopping the clock" or "turning back the clock."

THE WINNER'S CURSE: In a single-item auction with interdependent values, a bidder's expected value conditional on winning the item is less than her unconditional expected value.

Now consider, for example, the flat demands model with $M = 3$ and $\lambda = 2$. If the goods are assigned efficiently, then winning *one* unit indicates to a bidder that her signal equaled the *second-order* statistic of all bidders' signals, while winning *two* units indicates to a bidder that her signal equaled the *first-order* statistic of all bidders' signals. Winning more units is "worse news" than winning fewer units. (See also Ausubel and Cramton, 2002.) Thus, the flat demands model with assumptions (A.1)–(A.5) exhibits:

THE CHAMPION'S PLAGUE (OR GENERALIZED WINNER'S CURSE): In a multiple-item auction with interdependent values, a bidder's expected value conditional on winning a larger quantity is less than her expected value conditional on winning a smaller quantity.

Observe that the static auction does not allow bidders to account for the Champion's Plague in their bidding without impairing efficiency, while the dynamic auction does. We have:

THEOREM 4: Let the bidders have either identical capacities λ, where M/λ is *not* an integer, or unequal capacities λ_i. Then in a symmetric model with interdependent values satisfying (A.1)–(A.5), all equilibria of the Vickrey auction are inefficient. The turning-back-the-clock version of the alternative ascending-bid auction, however, possesses an efficient equilibrium. Moreover, if bidders' signals are independent, then the alternative ascending-bid auction raises strictly higher expected revenues than the Vickrey auction.

If bidders' signals are independent, then in the "flat demands" model studied in this section, revenue maximization uniquely coincides with efficiency (Ausubel and Cramton, 1999, Proposition 1). Consequently, the fourth sentence of Theorem 4 follows from the third sentence of Theorem 4. It is an open question whether this revenue ranking extends to flat demands in a symmetric model where signals are strictly affiliated.

At the same time, Theorem 4 should *not* be misinterpreted to suggest that there does not exist *any* efficient static mechanism in this environment. Indeed, in Appendix B of the working paper version (now a separate paper, Ausubel, 1999), as well as in Philippe Jehiel and Benny Moldovanu (2001) and Perry and Reny (2002), an efficient direct mechanism is derived. Rather, the correct interpretation of Theorem 4 is merely that the rules of the standard Vickrey auction do not allow efficiency in the face of value interdependencies.

V. Limitations of the Auction Design for Interdependent Values

While the properties of the proposed auction design are quite powerful for environments of private values, there are two basic limitations of the proposed auction design as applied to environments of interdependent values. First, the set of interdependent-values environments considered is quite limited. Second, the "turning-back-the-clock" rule required to treat the limited set of interdependent-values environments (but *not* needed for private-values environments) introduces possibilities for manipulation. This section discusses each of these two limitations.

The interdependent-values environments treated in Section IV are limited to those with symmetric bidders each possessing "flat demand curves." Each bidder i has a capacity λ_i and a constant marginal value V_i for all quantities in the interval $[0, \lambda_i]$. The constancy of marginal value in the interval $[0, \lambda_i]$ is common knowledge; the private information relates only to the *level* of V_i. This restriction excludes many interesting examples. For example, a bidder is not permitted to have a parameterized family of downward-sloping linear demand curves such as $V_i(q_i, \mathbf{S}) = v(\mathbf{S}) - q_i$, for $q_i \in [0, v(\mathbf{S})]$. The reason why such a family is excluded is as follows. If the bidding within this parameterized family of downward-sloping linear demand curves were fully separating, then opponents could infer the bidder's exact type based on the quantity she bid at price zero. Given that this bidder will continue to bid positive quantities at positive prices, however, she has incentive to distort her bid downward in order to reduce her opponents' beliefs and thereby win goods more cheaply. This incentive makes the equilibrium complex and inefficient.

The reason for limiting attention to bidders with flat demand curves is that such bidders have no incentive to distort in the way described in the previous paragraph. The bidder does not fully reveal her private signal until she drops out of the auction, and by then it is too late for the information to be used against her. This makes it possible to construct equilibria without informational distortion, and the clinching rule eliminates incentive for noninformational quantity distortion.

While this limitation on environments is unfortunately strong, it is not so severe as to render the interdependent-values results irrelevant. First, while confining, the family of environments treated here is nevertheless a strict generalization of the Milgrom and Weber (1982a) model, which continues to be used as the basis for most theoretical and empirical analyses of ascending-bid auctions with interdependent values. Second, while determining equilibria for bidders with interdependent values but strictly downward-sloping demand curves will be a formidable task—precisely because a fully efficient equilibrium does not exist—it still seems likely that the proposed dynamic-auction format will achieve greater efficiencies than the corresponding static auction. Third, it can be argued that even the "general" interdependent-values environments for which efficient direct mechanisms can be constructed are not all that general. They still require making strong assumptions such as that each bidder's signal is one-dimensional (see Maskin, 1992; Jehiel and Moldovanu, 2001), as well as value monotonicity and a single-crossing property (see Crémer and McLean, 1985; Ausubel, 1999).

The "turning-back-the-clock" rule that is required for the efficiency results with interdependent values introduces some possibilities for manipulation by bidders. While many forms of manipulative behavior are possible, the problem can be seen most easily by considering one very simple form of manipulation: a bidder's secret use of multiple bidding identities. This "shill bidding" problem, considered in Makoto Yokoo et al. (2004) and Ausubel and Milgrom (2002), may be viewed as an extreme form of collusion; a bidder secretly establishes multiple identities in the auction and bids them in a coordinated fashion. Consider an auction in which there is no limitation on the quantity that may be purchased by an individual bidder. Suppose that bidders are able to establish fictitious bidding identities without detection by the seller. A bidder in the "turning-back-the-clock" auction may elect to establish two fictitious identities and to enter all three identities (the actual bidder name and the two shill identities) into the auction. In the extreme case, the three identities each bid the maximum quantity until most or all opposing bidders drop out of the auction. The role of the first shill is to drop out and thereby guarantee that the auction restarts at a zero price. The role of the second shill is to drop out immediately thereafter, causing the actual bidder to clinch most or all of the units at essentially zero prices.

Note that this form of manipulation does not occur in the "stopping-the-clock" version of the auction. The possibility of inducing the price to restart is not present. Apart from the restarting possibility, bidding under a single identity maximizes the opportunities for a bidder to clinch units at low prices, so bidding under multiple identities would not help—and would likely harm—the bidder. More precisely, in a private values environment, and given the (weakly) diminishing marginal values assumed in this article, a bidder can never benefit from bidding under multiple identities under "stopping the clock." (This follows from the analysis of the static Vickrey auction by Yokoo et al., 2004; and Ausubel and Milgrom, 2002.) It is more difficult to make categorical statements under interdependent values—since bidders' expectations then become important—but it seems likely that if the multiple identities bid a higher quantity than the bidder would bid singly, this would cause opponents to make positive inferences about value, leading the opponents to stay longer in the auction and harming the manipulative bidder.

Given the possibilities for manipulation introduced by a "turning-back-the-clock" rule, it seems quite possible that an efficiency- or revenue-maximizing seller might pass up this device and settle instead for the "stopping-the-clock" version in which price never decreases.[32] Even for the interdependent-values environment with flat demand curves, the calculation of equilibrium under the "stopping-the-clock" rule is very difficult. One presumes that, while inferior to those obtained under the "turning-back-the-clock" rule, the "stopping-the-clock" equilibria are still probably superior to those of the sealed-bid multi-unit Vickrey auction.[33] Thus, the seller who elects to use the "stopping-the-clock" rule would likely retain some of the benefits associated with a dynamic auction, while greatly reducing the risk of manipulation.

VI. Conclusion

This article has proposed a new ascending-bid auction for multiple objects. This auction format occupies the analogous relationship with respect to the sealed-bid (multi-unit) Vickrey auction that the English auction occupies with respect to the sealed-bid second-price auction. In an environment where bidders have pure private values, the new dynamic auction game exhibits a sincere-bidding equilibrium that attains full efficiency and replicates the outcome of the classic Vickrey auction. For some formalizations of the auction game, the sincere equilibrium is the unique outcome of iterated weak dominance. In a symmetric environment where bidders have affiliated signals and interdependent values, the auction continues to possess an efficient equilibrium, whereas all equilibria of the standard sealed-bid auctions are inefficient.

Many of the advantages of dynamic auctions over static auctions that have been advanced elsewhere in the literature appear to apply to the auction format proposed here. When bidders' signals are affiliated in a symmetric environment, a dynamic auction may generate greater revenues than the analogous static auction (Milgrom and Weber, 1982a). When bidders' values are interdependent, a dynamic auction may allocate items more efficiently than the analogous static auction (Maskin, 1992). When the auction process otherwise fails to protect the confidentiality of bidders' valuations, a dynamic auction may enhance privacy preservation as compared to the analogous static auction (Rothkopf et al., 1990). When budget constraints impair the bidding of true valuations in a sealed-bid Vickrey auction, a dynamic auction may facilitate the expression of true valuations while staying within budget limits (Ausubel and Milgrom, 2002). All of these advantages are obtained in the current article while adhering to the fundamental prescription of making bidders' payments as independent as possible of their own reports (Vickrey, 1961).

Some other possible advantages of dynamic auctions over static auctions are difficult to model explicitly within standard economics or game-theory frameworks. For example, as emphasized in the introduction, it is generally held that the English auction is simpler for real-world bidders to understand than the sealed-bid second-price auction, leading the English auction to perform more closely to theory. One might expect this advantage to carry over to the comparison between the ascending-bid auction proposed here and the multi-unit Vickrey auction. The current article does not contain any formal analysis of this hypothesis, yet its validity is clearly important for a real-world seller deciding among alternative auction formats. Three sets of researchers, however, have recently run laboratory experiments involving the new dynamic auction, enabling us to begin obtaining some practical insights.

[32] Alternatively, as in the working paper version (Ausubel, 1997), the seller might utilize an intermediate version of the "turning-back-the-clock" rule. For example, after stopping at time t, the clock is restarted at a price of $p = 0.9 \sup_{t'<t}\{p(t')\}$. Thus, price is allowed to be rolled back somewhat, but not all the way back to zero. In order to make an effective choice of the constant, however, the auctioneer would need to know considerable information about the structure of bidders' signals and values.

[33] We know that all "stopping-the-clock" equilibria are inferior to the best "turning-back-the-clock" equilibrium, since the latter attains the optimum and the former (by reasoning similar to that of Theorem 4) cannot. Note that the intuition for the superior performance of the "turning-back-the-clock" auction is that it allows bidders to adjust dynamically for the Champion's Plague, lowering their bidding thresholds after winning units. By similar reasoning, it seems probable that the "stopping-the-clock" equilibria would be superior to those of the static Vickrey auction, since the "stopping-the-clock" format allows bidders to compensate partially for the Champion's Plague—they can reduce their bidding thresholds after winning some units, but not necessarily as far as they would like to—whereas the static Vickrey auction does not allow bidders to adjust dynamically for the Champion's Plague at all.

Kagel and Dan Levin (2001) and Kagel et al. (2003) have tested the proposed auction using a clever experimental design in which human subjects play against computers in a two-unit, private-values setting. Kagel and Levin report that more than 99 percent of the predicted gains from trade, and essentially exactly the predicted revenues, are realized. Kagel et al. compare the proposed auction with the Vickrey auction, finding greater allocative efficiency in the proposed auction:

> "The Ausubel auction comes significantly closer to sincere bidding than the static Vickrey auction even though the latter has a stronger solution concept (implementation in weakly dominated strategies versus iterated deletion of weakly dominated strategies). This suggests a tradeoff between the simplicity and transparency of a mechanism and the strength of its solution concept for less than fully rational agents" (Kagel et al., 2003).

Dirk Englemann and Veronika Grimm (2003) perform two-unit, private-values experiments using human subjects only, and they report very similar results. By contrast, Alejandro M. Manelli et al. (2000) run three-unit, interdependent-values experiments, obtaining higher revenues in the dynamic auction but greater efficiency in the static auction. As in the other experiments, their subjects understood that it was better to avoid bidding above their values in the dynamic than in the static auction. However, their subjects engaged in a different form of overbidding—bidding for all three units when they were told that they had value for only two units—and this accounted for the reversal in efficiency as compared to the other experiments.

There are at least two directions in which the results of this article can be extended. First, as discussed in Section V, the treatment here of interdependent values is very limited. Some other authors have begun to obtain more general results for interdependent values. Perry and Reny (2001) show that, with two bidders, my auction continues to yield efficient outcomes in a general specification of interdependent values based on one-dimensional signals. They further show that by expanding the procedure to allow bidders to submit *directed demands* (one against each other bidder), it is possible to obtain efficient outcomes with n bidders, identical objects, and interdependent values based on one-dimensional signals. Such extensions come at the cost of greater complexity for bidders and—similar to the critique of the "turning-back-the-clock" rule in Section V—they may introduce new possibilities for manipulation. Still, it seems likely that the ideas in this article can be extended to produce practical auction designs appropriate for richer informational environments than those treated here.

Second, the current article and most other recent work on efficient dynamic auctions of multiple items has restricted attention to homogeneous goods. Nevertheless, even in clear real-world examples of auctions of identical items (e.g., auctions of three-month Treasury bills), there often occur in close proximity other auctions of related but different goods (e.g., auctions of six-month Treasury bills). In Ausubel (2002), I expand the environment to allow bidders with concave utilities over heterogeneous commodities. Instead of a single price "clock," I utilize multiple independent clocks to generalize the auction procedure herein. Subject to a few caveats, I conclude there that it is still possible to replicate the outcome of the Vickrey-Clarke-Groves mechanism with a dynamic auction procedure.

Appendix

PROOF OF THEOREM 1:

At every point in the alternative ascending-bid auction up until its end, all of the payoff-relevant events in the auction occur through clinching. The cumulative quantity of clinched units for bidder i at time (and price) t is given by equations (2) and (3). Observe that the right side of equation (2) is independent of bidder i's actions; hence, changing one's own bid strategy can have no effect on payoff, except to the extent that: (i) it leads rival bidders to respond; or (ii) it determines one's own final quantity x_i^*.

Since marginal utilities were assumed (weakly) diminishing, the sincere bidding strategy given by equation (8) always yields monotonically nonincreasing quantities over time. Moreover, sincere bidding by bidder i always yields a final price p^* and final quantity x_i^* satisfying $x_i^* \in \arg\max_{x_i \in X_i}\{U_i(x_i) - p^*x_i\}$.

If all rival bidders $j \neq i$ bid sincerely, then rivals never respond to bidder i's strategy, except through price. Hence, sincere bidding is a mutual best response for every bidder—for every realization of utilities and after every history—and hence it is an ex post perfect equilibrium. Given the sincere bids of equation (8) and the payoff equation (7), it always yields the efficient outcome of the Vickrey auction. Moreover, with no bid information, a rival cannot distinguish between two strategies of bidder i, except to the extent that one or the other ends the auction. Hence, changing one's own bid strategy cannot lead rivals using *any* strategy to respond. We conclude that sincere bidding is a weakly dominant strategy in the auction with no bid information.

PROOF OF THEOREM 2:

First, we provide an order of elimination that yields sincere bidding as the outcome of iterated weak dominance. We begin with the last period, T. Since bidders' marginal valuations are assumed to be bounded by $\bar{u} < T$, each bidder wishes to minimize the number of units won at time T. Given that the rationing rules in case of over- or under-subscription are monotonically increasing in the final bids (see footnotes 17 and 18), the sincere bids $x_i^T = C_i^{T-1}$ weakly dominate all insincere bids $x_i^T > C_i^{T-1}$, and so all strategies specifying insincere bids at time T can be eliminated. Now suppose that with k iterations of weak dominance, all strategies other than sincere bidding have already been eliminated at times $t = T - k + 1, \ldots, T$. Then the choice of x_i^{T-k} can have no effect on the subsequent bidding by opponents. By the full support assumption, there is a positive probability that bidder i's opponents have valuations leading them each: (i) to bid this period in a neighborhood of their maximum allowable bids, i.e., $x_j^{T-k} > x_j^{T-k-1} - \varepsilon$, for all $j \neq i$; and (ii) to bid next period in a neighborhood of their minimum allowable bids, i.e., $x_j^{T-k+1} < C_j^{T-k} + \varepsilon$, for all $j \neq i$. If x_i^{T-k} is greater than bidder i's sincere demand, then in this event (and again using the assumption that the rationing rules are monotonically increasing in the final bids [see footnotes 17 and 18]), bidder i will unprofitably win units at time $T - k + 1$ that she could have avoided winning by bidding sincerely at time $T - k$. Hence, all strategies specifying bids greater than the sincere demand at time $T - k$ can be eliminated. Also by the full support assumption, there is a positive probability that bidder i's opponents have valuations leading them each to bid this period in a neighborhood of their minimum allowable bids, i.e., $x_j^{T-k} < C_j^{T-k-1} + \varepsilon$, for all $j \neq i$. If x_i^{T-k} is less than bidder i's sincere demand, then in this event (and also using the assumption that the rationing rules are monotonically increasing in the final bids), bidder i will unprofitably forego winning units in period T that she could have won by bidding sincerely. Hence, all strategies specifying bids less than the sincere demand at time $T - k$ can be eliminated. By induction, all strategies specifying insincere bids at any time and after any history of the auction can be eliminated in $(T + 1)$ iterations of weak dominance.

Second, we show that sincere bidding is the unique outcome of iterated weak dominance. First, sincere bidding is *never* eliminated. (Suppose otherwise. Consider the first round of elimination in which sincere bidding is eliminated for any type of any bidder, and let σ_i denote the strategy that dominates it. In this round, sincere bidding by all types of all bidders $-i$ is still possible. Against sincere bidding by bidders $-i$, and given full support, one can always choose a realization of types for bidders $-i$ such that sincere bidding yields a higher payoff for bidder i than does σ_i. This contradicts that sincere bidding could be eliminated.) Second, given this, and following the procedure of the previous paragraph, all strategies other than sincere bidding can be eliminated in $(T + 1)$ more iterations of weak dominance, independent of the eliminations that have already occurred.

AN EFFICIENT EQUILIBRIUM OF THE ALTERNATIVE ASCENDING-BID AUCTION FOR THE SYMMETRIC INTERDEPENDENT VALUES MODEL

An equilibrium will be constructed so that every bidder i bids up to her expected valuation, *conditional on the lowest of the other active bidders' signals equaling her own signal.* Upon reaching her expected valuation, she will drop to her lowest allowable quantity, $C_i(h)$. Since the bidding threshold is monotonically increasing in bidder i's signal s_i, for each equilibrium history h, bidders will drop out of the auction in (increasing) order of their signals s_i, as required for efficiency. Moreover, each bidder clinches units precisely in those situations where her expected value exceeds the clinching price, and she fails to clinch units precisely in those situations where her expected value is less than the clinching price, so each bidder maximizes her payoff against the opposing bidders' strategies.

For any bidder i ($i = 1, \ldots, n$), and for any j ($j = 1, \ldots, n - 1$), let Y_j^{-i} denote the j^{th}-order statistic of the signals received by all of the bidders excluding bidder i. Using the symmetry assumption A.4, the distribution of Y_j^{-i} is independent of i, and so the superscript "$-i$" will henceforth be suppressed from Y_j^{-i}. Given knowledge of the realizations of the order statistics $Y_j = y_j, \ldots, Y_{n-1} = y_{n-1}$, we may define:

$$(9)\quad w_j(s, y; y_j, \ldots, y_{n-1}) = E[V_i | S_i = s, Y_{j-1} = y, Y_j = y_j, \ldots, Y_{n-1} = y_{n-1}], \qquad \text{for } j = 2, \ldots, n.$$

Without knowledge of the realizations of the order statistics, we may define the corresponding value:

$$(10)\quad v_j(s, y) = E[V_i | S_i = s, Y_{j-1} = y], \qquad \text{for } j = 2, \ldots, n.$$

Finally, let us define a bidder i to be *active* after history h if and only if $x_i(h) > C_i(h)$. Furthermore, define $J(h) = |\{i \in N : x_i(h) > C_i(h)\}|$ to be the cardinality of the set of active bidders after history h. Then $n - J(h)$ bidders have dropped out at history h. Let bidder i be one of the remaining active bidders and suppose that the bidders who have dropped out correspond to the order statistics $Y_{J(h)} = y_{J(h)}, \ldots, Y_{n-1} = y_{n-1}$. We can define the equilibrium bidding threshold for any active bidder i to be her expected value for the objects, conditional on the lowest of the other active bidders' signals equaling her own signal s (and on the inferred realizations of $Y_{J(h)}, \ldots, Y_{n-1}$). Algebraically, this is expressed by:

$$(11)\quad \beta_i(s_i, h) = w_{J(h)}(s_i, s_i; y_{J(h)}, \ldots, y_{n-1}) \text{ and } \gamma_i(s_i, h) = C_i(h),$$

where the realizations $y_{J(h)}, \ldots, y_{n-1}$ may be inferred, inductively, from the history h and the equilibrium strategies. Note that no updated inference is drawn from an opponent reducing to $x_j > C_j(h)$.

An Efficient Equilibrium of the Vickrey Auction when $m = M/\lambda$ Is an Integer

By similar reasoning, in the Vickrey auction, with $m = M/\lambda$ an integer, the following bidding strategy for each bidder, for all quantities $q = 1, \ldots, \lambda$, and for all signals s_i, is an (efficient) equilibrium:

$$(12)\quad b_i^q(s_i) = v_{m+1}(s_i, s_i), \qquad \text{for } i = 1, \ldots, n.$$

The strategy of equation (12) is almost the same as the bidding threshold of equation (11) evaluated at $J(h) = m + 1$, except for the fact that the inferred realizations of $Y_{m+1}, \ldots, Y_{n-1}$ are unavailable in the Vickrey auction.

PROOF OF THEOREM 3:

If all n bidders use the strategies defined by equation (12) (and using the symmetry assumed in A.1 and A.4), the seller's expected revenues in the Vickrey auction are given by $E[v_{m+1}(Y_m, Y_m) | S_i > Y_m]$. Meanwhile, if $m = M/\lambda$ is an integer and if all n bidders use the strategies defined by equation (11), the seller's expected revenues in the alternative ascending-bid auction are given by $E[w_{m+1}(Y_m, Y_m; Y_{m+1}, \ldots, Y_{n-1}) | S_i > Y_m]$. Closely following Milgrom and Weber (1982a, Theorem 8), we now demonstrate that the first quantity is no greater than the second quantity. Observe that, if $s > y$, then:

$$\begin{aligned} v_{m+1}(y, y) &= E[V_i | S_i = y, Y_m = y] \\ &= E[E[V_i | S_i = s, Y_m = y, Y_{m+1} = y_{m+1}, \ldots, Y_{n-1} = y_{n-1}] | S_i = y, Y_m = y] \\ &= E[w_{m+1}(S_i, Y_m; Y_{m+1}, \ldots, Y_{n-1}) | S_i = y, Y_m = y] \\ &= E[w_{m+1}(Y_m, Y_m; Y_{m+1}, \ldots, Y_{n-1}) | S_i = y, Y_m = y] \\ &\le E[w_{m+1}(Y_m, Y_m; Y_{m+1}, \ldots, Y_{n-1}) | S_i = s, Y_m = y]. \end{aligned}$$

Consequently, taking the conditional expectation of each side of this inequality, given $S_i > Y_m$, yields:

$$\begin{aligned} E[v_{m+1}(Y_m, Y_m) | S_i > Y_m] &\le E[E[w_{m+1}(Y_m, Y_m; Y_{m+1}, \ldots, Y_{n-1}) | S_i, Y_m] | S_i > Y_m] \\ &= E[w_{m+1}(Y_m, Y_m; Y_{m+1}, \ldots, Y_{n-1}) | S_i > Y_m]. \end{aligned}$$

This inequality establishes that the seller's expected revenue from the static auction is no greater than from the dynamic auction, as required.

PROOF OF THEOREM 4:

Suppose, to the contrary, that there exists an ex post efficient equilibrium of the Vickrey auction, but that $\lambda_i \equiv \lambda$ ($i = 1, \ldots, n$) and M/λ is not an integer. Let m be the greatest integer such that $m\lambda < M$. By Lemma 1 of Ausubel and Cramton (2002), full efficiency requires that all bidders use the same flat bid function: there exists a strictly increasing function $\phi(\cdot)$ such that bidder i bids $b_i^q(s_i) = \phi(s_i)$, for all $i = 1, \ldots, n$, for all quantities $q = 1, \ldots, \lambda$, and for almost every signal s_i. If all bidders use the same flat bid function, however, bidder i's bid for her first unit is $b_i^1(s_i) = v_{m+2}(s_i, s_i)$, since she wins 1 unit if and only if her signal is at least the $(m + 1)^{\text{st}}$ order statistic of rivals' signals. Meanwhile, bidder i's bid for her last unit is $b_i^\lambda(s_i) = v_{m+1}(s_i, s_i)$, since she wins λ units if and only if her signal is at least the m^{th} order statistic of rivals' signals. Consequently, $b_i^\lambda(s_i) < b_i^1(s_i)$, contradicting the existence of an efficient equilibrium. If the λ_i are unequal, similar reasoning can be applied to a bidder with maximum λ_i.

Meanwhile, the strategies defined by equation (11) provide an ex post efficient equilibrium of the "turning-back-the-clock" version of the alternative ascending-bid auction.

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