An Efficient Dynamic Auction for Heterogeneous Commodities

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Abstract

This paper proposes a new dynamic design for auctioning multiple heterogeneous commodities, generalizing earlier work that treated identical objects. An auctioneer wishes to allocate one or more units of each of *K* heterogeneous commodities to *n* bidders. The auctioneer announces a vector of current prices, bidders report back quantities demanded at these prices, and the auctioneer adjusts the prices. Units are credited to bidders at the current prices as their opponents' demands decline, and the process continues until every commodity market clears.

Bidders, rather than being required to behave as price-takers, are permitted to strategically exercise their market power. Nevertheless, with pure private values, the proposed auction yields Walrasian equilibrium prices. An efficient outcome results, as from a Vickrey-Clarke-Groves mechanism, but bidders are only required to evaluate their demands along a one-dimensional path of prices, rather than reporting their utilities over the entire (*K*-dimensional) consumption set. The auction is also more transparent and potentially simpler for bidders to understand, and has the advantage of assuring the privacy of the upper portions of bidders' demands.

Theoretically, the new auction provides a new foundation for convergence to Walrasian equilibrium in an exchange economy. Practically, it provides an efficient method for simultaneously auctioning two or more related types of securities—for example, three-month and six-month Treasury bills. It is also potentially useful for auctioning multiple types of communications spectrum—for example, licenses for paired and unpaired spectrum.

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An Efficient Dynamic Auction for Heterogeneous Commodities

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In earlier work (Ausubel, 1997, 2000a), I proposed an efficient ascending-bid auction design for multiple objects. In environments with homogeneous goods, where bidders have pure private values and diminishing marginal values, this dynamic auction yields outcomes coinciding with that of the (sealed-bid) Vickrey (1961) auction, but offers advantages of transparency, simplicity and privacy preservation. Moreover, in some environments where bidders have interdependent values for the objects, this dynamic auction continues to yield efficient outcomes and thus outperforms even the Vickrey auction.

However, situations abound in diverse industries in which heterogeneous (but related) commodities are auctioned. On a typical Monday, the U.S. Treasury sells some $8 billion in three-month bills and $5 billion in six-month bills.[1] Current practice is to auction the three-month and six-month bills separately in two independent sealed-bid auctions. In the European UMTS/IMT-2000 spectrum auctions, governments are selling both paired and unpaired 3G spectrum, located at similar frequencies but apparently exhibiting markedly different values. Some governments are selling these in fixed bundles, while others are selling the paired spectrum followed by the (less valuable) unpaired spectrum. In the United Kingdom gas industry, auctions have recently been conducted for capacity at several entry points into the gas pipeline system. Entry rights at different geographic locations are substitutes and can equally be viewed as heterogeneous (but related) commodities.

The current paper proposes to extend my auction design for homogeneous goods to the case of K heterogeneous commodities. In the same spirit as the earlier work, it attempts to do this in the simplest possible way. As before, the auctioneer iteratively reports prices to bidders, and bidders respond with quantities that they desire to transact. Units are credited to bidders at current prices when it appears that they are no longer desired by opponents, and the process continues until every commodity market clears.

However, the extension of an efficient dynamic auction to $K \geq 2$ commodities poses at least two significant obstacles. First, unlike in the homogeneous goods case, a bidder may now wish to increase her demand for a given commodity along the path toward equilibrium, as prices of substitute commodities

[1] On Monday 3 July 2000, the U.S. Treasury auctioned $8.412 billion in three-month bills and $4.744 billion in six-month bills (Press Release, Department of the Treasury, Bureau of the Public Debt, web-posted at http://www.publicdebt.treas.gov).

increase. Thus, units that once appeared to be "clinched" by another bidder may later be "unclinched", and the auction rules need to reflect this. Second, *K* simultaneous auctions are effectively conducted, and it is unclear how the progress of one auction should affect the clinching of units in another. Surprisingly, this paper establishes that it suffices to independently calculate the crediting of different commodities; the only formal interaction among the *K* auctions needs to occur through the simultaneous bidding and the price adjustment rule. With these obstacles resolved, an efficient dynamic auction design for heterogeneous commodities emerges.

Besides my earlier work proposing the auction design for homogeneous goods, the closest related literature includes other papers that seek to adapt the basic idea to more general environments. Perry and Reny (1999b) adapt the homogeneous goods design to more general environments with interdependent values, while Ausubel (1996) makes a first attempt to adapt the idea to the case of dissimilar objects. Bikhchandani and Ostroy (2000) demonstrate that the homogeneous goods design is a primal-dual algorithm applied to an appropriate linear programming formulation and, in a recent presentation, they report a possible extension to heterogeneous objects which appears to be significantly different from—but related to—the current proposed design. Other related literature include the early work of Kelso and Crawford (1982) on efficient matching, as well as recent work by Gul and Stacchetti (1999, 2000) and Milgrom (2000) which analyzes similar problems but assumes that bidders behave nonstrategically. Also related is the literature focused on the possibility or impossibility of efficient auction design when bidders have interdependent values, which includes Dasgupta and Maskin (2000), Jehiel and Moldovanu (1999), Maskin (1992) and Perry and Reny (1999a,b), as well as Ausubel (2000b).

The current paper also connects with the venerable literature on tâtonnement stability and adjustment processes, which seeks to understand the forces operating in an economy that may drive it toward an equilibrium (see, for example, the survey by Hahn, 1982). The most famous early attempt to treat convergence to equilibrium was made by Walras (1874). A number of relatively recent contributions are cited in Section 5, below. Most of this literature makes the assumption of perfect competition, which—apart from its usual weaknesses—produces in dynamic models "the paradoxical problem that a perfect competitor changes prices that he is supposed to take as given" (Arrow and Hahn, 1971, p. 322). The current paper suggests a different approach to these issues. Instead of price-taking agents interacting via the fictitious Walrasian auctioneer, optimizing bidders interact through an explicit auction procedure. Despite their strategic interaction, they drive prices to a Walrasian equilibrium.

The main results of the current paper are the specification of the auction procedure and the proof that it works. Bidders' preferences are required to be quasilinear with respect to money, but otherwise only standard assumptions from the general equilibrium literature are made. If a bidder's opponents all bid sincerely, then the bidder's payoff is path-independent and equals a constant translation of the total social surplus at the final allocation (Lemma 2). Hence, if any Walrasian price vector is attainable, the bidder maximizes her payoff by choosing to end at a Walrasian equilibrium (Theorem 1).

A stable price adjustment combination is defined to consist of a price adjustment process, assumptions on bidders' preferences and an initial condition, which together guarantee convergence to Walrasian equilibrium for a competitive economy. The same combination yields sincere bidding as an equilibrium of the new auction design, with the outcome that of a Vickrey auction with a reserve price (Theorem 2). With appropriate choice of the initial price, it yields exactly the Vickrey auction payoff to a single bidder (Theorem 3), and an (n+1)-step procedure is provided which yields exactly the Vickrey auction payoff to all n bidders (Theorem 4).

Suppose that we specialize our focus to an environment with gross substitutability, in the sense that if the prices of some commodities are increased while the prices of all other commodities are held constant, then a bidder's demand weakly increases for each of the commodities whose prices were held constant. Coupled with any continuous and sign-preserving transformation of the Walrasian tâtonnement process, the proposed design has an equilibrium in which bidders bid sincerely and the price ascends[2] to the unique Walrasian equilibrium[3] (Corollary 1). Thus, information about bidders' valuations at prices above the equilibrium is never elicited, and so the auction design exhibits the same advantage of privacy preservation as in the homogeneous goods case.

1. An Illustration of the Main Ideas

Suppose that a seller wishes to allocate units of each of K types of heterogeneous commodities among n bidders, by way of a simultaneous dynamic auction. The simplest dynamic method that one might consider for auctioning the K types of commodities is a "uniform-price, ascending-bid auction." The seller announces a vector of current prices, bidders report back quantities demanded at these prices, and the

[2] This assumes that we are considering an auction to sell the heterogeneous commodities, and that we begin at a sufficiently low initial price. If we consider a "reverse" or "procurement" auction to buy, and if the heterogeneous commodities are substitutes in production, then the analogous procedure started at a sufficiently high price yields descending prices to an efficient outcome.

[3] Thus, the result closely draws upon the earlier work of Gul and Stacchetti (2000) and Milgrom (2000), but I do not need to assume that bidders bid "straightforwardly".

seller adjusts the prices. The process continues until every commodity market clears and then, in the spirit of a tâtonnement process, bidders receive the final quantities that they demanded and pay the final prices.

Unfortunately, both in theory and practice, the simple uniform-price, ascending-bid auction procedure typically does not yield Walrasian outcomes, when bidders—rather than being required to behave as price-takers—are permitted to strategically exercise their market power. This is most straightforwardly seen in the case of $K = 1$, i.e., in markets for homogeneous goods. Theoretically, when bidders possess private information about their valuations, every equilibrium of a uniform-price auction yields inefficient outcomes with positive probability (Ausubel and Cramton, 1996; Bolle, 1997; and Engelbrecht-Wiggans and Kahn, 1998). The reason for inefficiency is that uniform-price auctions create strong incentives for "demand reduction": a bidder will bid less than her valuation for a marginal unit, in order to depress the price that she pays for inframarginal units.

More extreme results are possible if the auction is explicitly sequential. Ausubel and Schwartz (1999) show that an alternating-bid version of the uniform-price ascending-bid auction for a divisible commodity possesses a unique subgame-perfect equilibrium. The first bidder bids the opening price on slightly more than half the units, the second bidder bids the next possible price on the remaining units, and the game ends. Thus, the allocation need not bear any connection with the efficient outcome, and the price (essentially the lowest allowable price) need not bear any connection to the Walrasian price. While the Ausubel and Schwartz result is admittedly extreme, the prediction was borne out empirically in the October 1999 German auction of extra telecommunications spectrum. As recounted by Jehiel and Moldovanu (2000), ten licenses of virtually homogeneous spectrum were offered to the four German mobile phone incumbents. In the first round of bidding, Mannesmann placed high bids of DM 36,360,000 per MHz on each of licenses 1–5 and of DM 40,000,000 per MHz on each of licenses 6–10. In the second round of bidding, T-Mobil raised Mannesmann on licenses 1–5 by bidding a price of 40,010,000 DM (the minimum bid increment was 10%), while letting Mannesmann maintain the high bid on licenses 6–10. In the third round of bidding, no new bids were entered, and so the auction ended with the two largest incumbents dividing the market at an apparently low price.

With $K = 1$, Walrasian outcomes can be obtained in an ascending-bid auction by using the procedure proposed in Ausubel (1997, 2000a). The seller announces a current price, bidders report back quantities demanded at this price, and the seller raises the price. The process continues until the market clears, but units may be awarded to bidders before the close, at the prices at which they are "clinched". The current paper shows that this procedure can be generalized, in a simple way, to the case where $K > 1$.

We conclude this section by demonstrating the new procedure for an example where $K = 2$. Real-world examples fitting this description may include the sale of three-month and six-month Treasury bills, or the sale of paired and unpaired telecommunications spectrum. However, we will generically refer to them as commodity A and commodity B. Suppose that the supply vector is (10,8), i.e., commodities A and B are available in supplies of 10 and 8, respectively, and suppose that there are $n = 3$ bidders. The auctioneer initially announces a price vector of $p_1 = (3,4)$, and subsequently adjusts the price vector to $p_2 = (4,5)$, $p_3 = (5,7)$, $p_4 = (6,7)$, and finally $p_5 = (7,8)$. The bidders' reports of quantities demanded at these price vectors are shown in Table 1:

Price Vector	Bidder 1	Bidder 2	Bidder 3
$p_1 = (3,4)$	(5,4)	(5,4)	(5,4)
$p_2 = (4,5)$	(4,4)	(5,4)	(4,3)
$p_3 = (5,7)$	(4,3)	(4,4)	(4,1)
$p_4 = (6,7)$	(4,3)	(4,4)	(3,2)
$p_5 = (7,8)$	(4,2)	(3,4)	(3,2)

Table 1: Price and Quantity Vectors for Illustrative Example with $K = 2$

The *crediting* of units to bidders occurs as follows. First, consider Bidder 1. When the price vector advances from $p_1 = (3,4)$ to $p_2 = (4,5)$, the sum of the quantity vectors demanded by Bidder 1's opponents decreases from (10,8) to (9,7). Thus, 1 unit of commodity A and 1 unit of commodity B can be thought of as becoming available to Bidder 1 at the current price of $p_2 = (4,5)$. The auction algorithm takes this literally, by *crediting* 1 unit of commodity A at a price of 4, and 1 unit of commodity B at a price of 5, to Bidder 1. Next, consider Bidder 2. When the price vector advances from $p_1 = (3,4)$ to $p_2 = (4,5)$, the sum of the quantity vectors demanded by Bidder 2's opponents decreases from (10,8) to (8,7). Thus, 2 units of commodity A and 1 unit of commodity B can be thought of as becoming available to Bidder 2 at the current price. The auction algorithm takes this literally, by *crediting* 2 units of commodity A at a price of 4, and 1 unit of commodity B at a price of 5, to Bidder 2. Finally, consider Bidder 3. When the price vector advances from $p_1 = (3,4)$ to $p_2 = (4,5)$, the sum of the quantity vectors demanded by Bidder 3's opponents decreases from (10,8) to (9,8). Thus, 1 unit of commodity A and 0 units of commodity B can be thought of as becoming available to Bidder 3 at the current price. Again, the auction algorithm takes this literally, by *crediting* 1 unit of commodity A at a price of 4, and 0 units of commodity B at a price of 5, to Bidder 3.

The process continues as the price vector advances. One interesting moment occurs when the price advances from p_3 = (5,7) to p_4 = (6,7). Observe that Bidder 3's demand vector changes from (4,1) to (3,2), while the other bidders' demand vectors remain constant. In particular, Bidder 3's demand for commodity B *increases*, meaning that 1 *fewer* unit of commodity B remains available for Bidders 1 and 2. Consequently, the auction algorithm needs to take this literally, by *debiting* 1 unit of commodity B at the current price of 7 from each of Bidders 2 and 3.

The entire progression of units credited and debited is summarized in Table 2:

Price Vector	Bidder 1	Bidder 2	Bidder 3
P_1 = (3,4)	Initialization	Initialization	Initialization
P_2 = (4,5)	1 unit of A credited at 4 1 unit of B credited at 5 Cumulative payment = 9	2 units of A credited at 4 1 unit of B credited at 5 Cumulative payment = 13	1 unit of A credited at 4 0 units of B credited at 5 Cumulative payment = 4
P_3 = (5,7)	1 unit of A credited at 5 2 units of B credited at 7 Cumulative payment = 28	0 units of A credited at 5 3 units of B credited at 7 Cumulative payment = 34	1 unit of A credited at 5 1 unit of B credited at 7 Cumulative payment = 16
P_4 = (6,7)	1 unit of A credited at 6 1 unit of B *debited* at 7 Cumulative payment = 27	1 unit of A credited at 6 1 unit of B *debited* at 7 Cumulative payment = 33	0 units of A credited at 6 0 units of B credited at 7 Cumulative payment = 16
P_5 = (7,8)	1 unit of A credited at 7 0 units of B credited at 8 Cumulative payment = 34	0 units of A credited at 7 1 unit of B credited at 8 Cumulative payment = 41	1 unit of A credited at 7 1 unit of B credited at 8 Cumulative payment = 31

Table 2: Credits and Debits for Illustrative Example with $K = 2$

At p_5 = (7,8), supply and demand are now in balance for both commodities. Thus, p_5 becomes the final price. Bidders 1, 2 and 3 receive their quantity vectors of (4,2), (3,4) and (3,2), respectively, demanded at the final price. Observe that, for each bidder, the quantity vector demanded at the final price equals the sum of all units credited or debited along the way. However, since many of the credits and debits occurred at earlier prices, bidders' payments do *not* generally equal their final demands evaluated at the final prices. Rather, if the procedure described above is performed along a continuous price path, the bidders' payments are those derived from a Vickrey auction (with a reserve price equaling the initial price vector). We develop this result in the following sections.

2. THE MODEL

A seller wishes to allocate units of each of K heterogeneous commodities among n bidders, $N \equiv \{1, \ldots, n\}$. The seller's available supply will be denoted by $S = (S^1, \ldots, S^K) \in \mathbb{R}^K_{++}$. Bidder i's consumption vector will be denoted by $x_i = (x_i^1, \ldots, x_i^K) \in X_i$, where the consumption set X_i is a compact subset of $\mathbb{R}^K_+$. Bidders are assumed to have pure private values for the commodities and to possess utilities that are quasilinear with respect to money. Bidder i's value is given by the function $U_i : X_i \to \mathbb{R}$. The functions $U_i(x_i)$ are assumed to be continuous, concave and strictly increasing. Bidder i's net utility equals her value for the consumption vector x_i less the payment y_i that she is required to make: $U_i(x_i) - y_i$.

The price vector will be denoted by $p = (p^1, \ldots, p^K) \in \mathbb{R}^K_+$. Bidder i's *indirect net utility function*, $V_i(p)$, and true *demand correspondence*, $Q_i(p)$, are defined respectively by:

$$V_i(p) = \max_{x_i \in X_i} \{U_i(x_i) - p \cdot x_i\}, \tag{1}$$

and $$Q_i(p) = \{x_i \in X_i : U_i(x_i) - p \cdot x_i = V_i(p)\}. \tag{2}$$

Observe that $V_i(p)$ is well defined and $Q_i(p)$ is nonempty. Since $U_i(\bullet)$ is continuous and concave, its conjugate function, $-V_i(\bullet) : \mathbb{R}^K_+ \to \mathbb{R}$, is continuous, closed and concave (Rockafellar, 1970, Thm. 12.2 and p. 308). Moreover, let $q_i(\bullet)$ be any[4] measurable selection from the correspondence $Q_i(\bullet)$. We have:

$$-q_i(p) \in \partial V_i(p), \text{ for all } p \in \mathbb{R}^K_+, \tag{3}$$

i.e., the negative of any true demand vector of bidder i at p is a subgradient of V_i at p (Rockafellar, 1970, Thm. 23.5). This is merely the non-differentiable version of the standard result that $\nabla V_i(p) = -q_i(p)$.

A *Walrasian equilibrium* is a price vector p^* and a consumption vector $\{x_i^*\}_{i=1}^n$ for every bidder such that $x_i^* \in Q_i(p^*)$, for $i = 1, \ldots, n$, and $\sum_{i=1}^n x_i^* = S$.

At most points in this paper, we will want the price vector to be interpretable as the prices in an auction process, in response to which the bidders submit bids of quantities. Let T be a finite positive time. With every time $t \in [0,T]$, we associate a price vector $p(t)$. The price vector is required to be a continuous function of time. At every time t, each bidder i selects a bid $x_i(t)$. Furthermore, we say that bidder i bids *sincerely* if her bid belongs to her true demand correspondence:

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[4] A judicious selection from $Q_i(\bullet)$ may be required at a Walrasian price vector w, in order that the expressed aggregate demand be equal to supply at w.

Sincere Bidding. Bidder i is said to bid sincerely if $x_i(t) = q_i(p(t))$, for all $t \in [0,T]$, where the function $q_i(\bullet)$ is a measurable selection from the demand correspondence $Q_i(\bullet)$.

In some examples, we will assume that each bidder's preferences satisfy a gross substitutability condition. An assumption of gross substitutability guarantees that a Walrasian tâtonnement process converges to a Walrasian equilibrium (Arrow, Block and Hurwicz, 1959) and enables an ascending auction process to readily converge to an efficient outcome (see Kelso and Crawford, 1982; Gul and Stacchetti, 1999, 2000; and Milgrom, 2000). When needed, we assume:

Gross Substitutability. $U_i(x_i)$ satisfies gross substitutability if for any two price vectors p and p' such that $p' \geq p$ and for any $x_i \in Q_i(p)$, there exists $x_i' \in Q_i(p')$ such that $x_i'^k \geq x_i^k$ for any commodity k such that $p'^k = p^k$.

Gross substitutability requires that if the prices of some commodities are increased while the prices of the remaining commodities are held constant, then a bidder's demand weakly increases for each of the commodities whose prices were held constant.

3. "CLINCHING" VERSUS "CREDITING AND DEBITING"

In Ausubel (1997, 2000a), I introduced the notion of "clinching" for auctions of homogeneous goods. In the current notation, this corresponds to the case of $K = 1$, and so our various quantity and price vectors temporarily reduce to scalars. Let:

$$x_{-i}(t) = \sum_{j \neq i} x_j(t)\,,\ \ \overline{c}_i(t) = \max\{0\,, S - x_{-i}(t)\} \text{ and } c_i(t) = \sup_{\overline{t} \in [0,t]} \overline{c}_i(\overline{t})\,, \tag{4}$$

and define the payment, $y_i(T)$, of bidder i by the following Riemann-Stieltjes integral:

$$y_i(T) = \int_0^T p(t)\, dC_i(t)\,. \tag{5}$$

Eq. (5) has the simple interpretation that, every time it becomes a foregone conclusion that bidder i will win additional units of the homogeneous good, she wins them at the current price. Suppose that an ascending-clock auction for homogeneous goods is initiated at $p(0) = 0$ and is allowed to run until such time T that the market clears. If bidder i is assigned $x_i(T)$ units and is assessed a payment $y_i(T)$ determined by Eqs. (4) and (5), while all bidders $j \neq i$ are bidding sincerely, then bidder i receives the same outcome as in the Vickrey (1961) auction. As a result, sincere bidding by every bidder is an (efficient) equilibrium of this ascending-bid auction for homogeneous goods.

It is also possible to modify Eq. (5) in a relatively innocuous way. Define instead the payment, $a_i(T)$, of bidder i by the following Riemann-Stieltjes integral:

$$a_i(T) = p(0)\left[S - x_{-i}(0)\right] - \int_0^T p(t)\, dx_{-i}(t)\,. \tag{6}$$

Suppose that $x_{-i}(t)$ is (weakly) monotonic. If $p(0) = p_{-i}$, where p_{-i} is defined implicitly by $\sum_{j \neq i} q_j(p_{-i}) = S$, then the payment $a_i(T)$ determined by Eq. (6) coincides with the payment $y_i(T)$ determined by Eq. (5). For other choices of $p(0)$, Eq. (6) has the interpretation of assessing bidder i the payment of a *Vickrey auction with reserve price* $p(0)$ (see Ausubel and Cramton, 1999). For example, suppose that $p(0) > p_{-i}$. Then, Eq. (6) awards surplus to bidder i in the spirit of the Vickrey auction. Calculate: (a) the social surplus in the efficient allocation if bidder i is present; and (b) the social surplus in the efficient allocation if bidder i is absent, but every unit of the good is treated as having an opportunity cost of at least $p(0)$. Then, in equilibrium, Eq. (6) awards bidder i exactly the difference between these two calculations. Essentially the same interpretation also applies if $p(0) < p_{-i}$; however, then the reserve price $p(0)$ is effectively applied to a negative quantity. Vickrey auctions with reserve prices are also incentive compatible. They are efficient in the sense that all units of the good which are distributed to bidders are assigned efficiently; however, they are inefficient in the sense that the aggregate quantity which is distributed to bidders may be less than efficient, due to the reserve price (see Ausubel and Cramton, 1999).

The principal difference between the original notion of clinching in Eqs. (4)–(5) and the extended notion in Eq. (6) occurs when $x_{-i}(t)$ is non-monotonic. The former only allows "clinching" to occur, while the latter—which drops any supremum-over-time term—allows both "crediting" and "debiting" to occur. The latter will be needed for Lemma 2 *et seq*. to hold.

4. THE EXTENSION TO K HETEROGENEOUS COMMODITIES

The most naïve way that one might think about generalizing the homogeneous goods procedure to the case of K heterogeneous commodities is to run K ascending clocks (one for each commodity) simultaneously and to independently compute "credits" and "debits" for each. Let the movement of the K ascending clocks be described by a continuous, piecewise smooth,[5] vector-valued function $p(t) = (p^1(t), \ldots, p^K(t))$ from $[0,T]$ to $\mathbb{R}_+^K$. Further suppose that each bidder i bids according to the vector-valued function $x_i(t) = \left(x_i^1(t), \ldots, x_i^K(t)\right)$ from $[0,T]$ to X_i, which is constrained to be of bounded

[5] The (vector-valued) continuous function p is said to be piecewise smooth if each component p^k has a bounded derivative which is continuous everywhere in $[0,T]$, except (possibly) at a finite number of points. At these exceptional points it is required that both right- and left-hand derivatives exist. A curve Γ is said to be piecewise smooth if it can be described by a piecewise smooth function (Apostol, 1957, Definition 9–61).

variation. Then the naïve extension of Eq. (6) would be to define $x_{-i}^k(t) = \sum_{j \neq i} x_j^k(t)$, for $k = 1, \ldots, K$, and to define payments by:

$$a_i(T) = p(0) \cdot [S - x_{-i}(0)] - \int_0^T p(t) \cdot dx_{-i}(t) \equiv \sum_{k=1}^K \left\{ p^k(0)\,[S^k - x_{-i}^k(0)] - \int_0^T p^k(t)\, dx_{-i}^k(t) \right\}, \qquad (7)$$

where each of the right-most integrals of Eq. (7) is calculated as a Riemann-Stieltjes integral. We begin by observing:

LEMMA 1. *The payment* $a_i(T)$ *of Eq.* (7) *is well defined.*

PROOF. By Theorem 9–26 of Apostol (1957), since p^k is continuous on $[0,T]$ and each x_j^k (and, hence, x_{-i}^k) is of bounded variation on $[0,T]$, each Riemann-Stieltjes integral $\int_0^T p^k(t)\, dx_{-i}^k(t)$ exists.■

If the price path is confined to the interior of its domain, and if each opponent $j \neq i$ of bidder i bids sincerely, we have:

LEMMA 2. *If the price p(t) is any piecewise smooth function from* $[0,T]$ *to* $\mathbb{R}_{++}^K$ *and if each bidder* $j \neq i$ *bids sincerely (i.e.,* $x_j(t) = q_j(p(t))$*, for all* $j \neq i$ *and for all* $t \in [0,T]$*), then the integral* $\int_0^T p(t) \cdot dx_{-i}(t)$ *in Eq.* (7) *is independent of the path from p*(0) *to p*(*T*) *and equals:*

$$\int_0^T p(t) \cdot dq_{-i}(p(t)) \;=\; U_{-i}(q_{-i}(p(T))) - U_{-i}(q_{-i}(p(0))) \;\equiv \sum_{j \neq i} \left[U_j(q_j(p(T))) - U_j(q_j(p(0))) \right]. \qquad (8)$$

PROOF. The Riemann-Stieltjes integral $\int_0^T p^k(t)\, dx_j^k(t)$ exists if and only if the Riemann-Stieltjes integral $\int_0^T x_j^k(t)\, dp^k(t)$ exists. Consequently, by integration by parts, and with sincere bidding:

$$\int_0^T p(t) \cdot dq_j(p(t)) = p(T) \cdot q_j(p(T)) - p(0) \cdot q_j(p(0)) - \int_0^T q_j(p(t)) \cdot dp(t), \text{ for all } j \in N. \qquad (9)$$

Let $\Gamma(0,T)$ denote the (piecewise smooth) curve in $\mathbb{R}_{++}^K$ described by $p(t)$. The integral on the right side of Eq. (9) may be rewritten as the line integral $\int_{\Gamma(0,T)} q_j \cdot dp$. (For a formal definition of the line integral, see Apostol, 1957, Definition 10–32.) Since $V_j(\bullet)$ is a convex function and $-q_j(\bullet)$ is a measurable selection from its subdifferential (see Eq. (3), above, and the surrounding text), Theorem 1 of Krishna and Maenner (2000) guarantees that the line integral is independent of path and equals $\int_{\Gamma(0,T)} q_j \cdot dp = -V_j(p(T)) + V_j(p(0))$. Noting that $V_j(p(t)) = U_j(q_j(t)) - p(t) \cdot q_j(p(t))$:

$$\int_0^T p(t) \cdot dq_j(p(t)) = U_j(q_j(p(T))) - U_j(q_j(p(0))), \qquad (10)$$

and summing over all $j \neq i$ yields Eq. (8). ■

In light of the path independence established by Lemma 2, the strategic choice by bidder i reduces from an optimization problem over price paths in $\mathbb{R}^K_{++}$ to an optimization problem over endpoints in $\mathbb{R}^K_{++}$. In the next definition, we let $\mathcal{P}_i$ denote the set of endpoints over which bidder i may choose:

DEFINITION 1. *The* ***set of all final prices attainable by i****, denoted $\mathcal{P}_i$, is the set of all prices at which the auction may terminate, given that all bidders $j \neq i$ bid sincerely, the specified price adjustment process, and all constraints on the strategy of bidder i. Furthermore, any attainable final price $p \in \mathcal{P}_i$ implies an* ***associated allocation*** *consisting of: $q_j(p)$, for each bidder $j \neq i$; and $S - q_{-i}(p)$, for bidder i.*

We then have:

THEOREM 1. *If each bidder $j \neq i$ bids sincerely and if bidder i's bidding is constrained so as to generate piecewise smooth price paths from $[0,T]$ to $\mathbb{R}^K_{++}$, then bidder i maximizes her payoff by maximizing social surplus over allocations associated with $\mathcal{P}_i$. Moreover, if a Walrasian price vector w is attainable by bidder i (i.e., if $w \in \mathcal{P}_i$), then bidder i maximizes her payoff by selecting a Walrasian price vector, and thereby receives her payoff from a Vickrey auction with a reserve price of $p(0)$.*

PROOF. By Lemma 2, bidder i's payoff from any bidding strategy $x_i(t)$ that generates piecewise smooth price functions from $[0,T]$ to $\mathbb{R}^K_{++}$ and concludes with bidder i receiving quantity $x_i(T)$ and with the market price equaling $p(T)$ is:

$$\begin{aligned} U_i(x_i(T)) - a_i(T) &= U_i(x_i(T)) - p(0)\cdot[S - x_{-i}(0)] + \int_0^T p(t)\cdot dq_{-i}(t) \\ &= U_i(x_i(T)) - p(0)\cdot[S - x_{-i}(0)] + U_{-i}(q_{-i}(p(T))) - U_{-i}(q_{-i}(p(0))). \end{aligned} \tag{11}$$

By definition, $p(T) \in \mathcal{P}_i$ and has associated allocation of $q_j(p)$, for each bidder $j \neq i$, and $S - q_{-i}(p(T))$ for bidder i. Consequently, bidder i receives payoff of:

$$U_i\left(S - q_{-i}(p(T))\right) + U_{-i}\left(q_{-i}(p(T))\right) - \left\{p(0)\cdot[S - q_{-i}(p(0))] + U_{-i}\left(q_{-i}(p(0))\right)\right\}. \tag{12}$$

Since the expression in braces within Expression (12)—determined only by the starting price and other bidders' starting actions—is a constant, bidder i maximizes Expression (12) by maximizing the first two terms. These first two terms coincide with social surplus for the allocation associated with $p(T)$.

Moreover, given the quasilinearity of utility, the Fundamental Theorems of Welfare Economics imply that any Walrasian equilibria is associated with a surplus-maximizing allocation, and vice versa. Consequently, if $\mathcal{P}_i$ contains a Walrasian equilibrium w, then bidder i maximizes social surplus over $\mathcal{P}_i$ by selecting some Walrasian equilibrium as the final price. Expression (12) then evaluates to yielding bidder i

the incremental surplus she brings to the auction if all units are evaluated at an opportunity cost of $p(0)$. Thus, bidder i receives the same payoff as from a Vickrey auction with reserve price of $p(0)$. ■

5. EQUILIBRIUM OF THE DYNAMIC AUCTION

In this section, we will take as our hypothesis any adjustment process (A), any assumption (B) on bidders' preferences, and any criterion (C) for stability (e.g., global stability or local stability) such that—in the classical setting of competitive equilibrium—the adjustment process (A) coupled with preference assumptions (B) guarantee convergence according to criterion (C) to Walrasian equilibrium, along a piecewise smooth price path. We will demonstrate that the identical combination of (A), (B) and (C) guarantees that sincere bidding is an (efficient) equilibrium of the new auction game. The result will allow us to import the large existing literature on stability of price adjustment processes to enumerate a variety of combinations of assumptions under which the new auction may achieve full efficiency.

DEFINITION 2. *The triplet* (A)–(B)–(C) *will be said to be a* ***stable price adjustment combination*** *for competitive economies if:*

(A) *is a price adjustment process,*
(B) *is a set of assumptions on bidders' preferences, and*
(C) *is a condition on the initial price for convergence (e.g., local, global or universal stability);*

and price adjustment process (A) *for an economy satisfying bidder assumptions* (B) *is guaranteed to converge to a Walrasian equilibrium along a piecewise smooth path in accord with initial condition* (C).

EXAMPLE 1. Let $Z(p(t)) = -S + \sum_{i=1}^{n} q_i(p(t))$ denote the vector of excess demands at time t and let (A) be a continuous and sign-preserving transformation $H^k(\bullet)$ of the Walrasian tâtonnement process:

$$\dot{p}^k(t) = H^k\left(Z^k(p(t))\right), \text{ for } k = 1, \ldots, K. \tag{13}$$

Let (B) include the assumption of gross substitutability plus additional assumptions on the economy guaranteeing that the excess demand for each commodity is a continuous function and that a positive Walrasian price vector exists. Let (C) be the condition of global convergence. Then (A)–(B)–(C) is a stable price adjustment combination (Arrow, Block and Hurwicz, 1959, Theorem 1).

EXAMPLE 2. Let (A) be the Global Newton price adjustment process. Let (B) consist of assumptions on the economy guaranteeing that the excess demand for each commodity is a C^2 function satisfying an extended boundary condition. Let (C) be the condition of convergence from all boundary points of $\mathbb{R}_{+}^{K}$, except for a set of measure zero. Then (A)–(B)–(C) is a stable price adjustment combination (Smale, 1976, Theorem C).

EXAMPLE 3. Let (A) be the price adjustment process proposed in Van der Laan and Talman (1987). Let (B) consist of assumptions on the economy restricting consumption vectors to the set $\mathbb{R}_{++}^{K}$ and guaranteeing that the excess demand function for each commodity is C^2 in the interior of $\mathbb{R}_{+}^{K}$. Let (C) be the condition of generic convergence starting from all points of $\mathbb{R}_{++}^{K}$. Then (A)–(B)–(C) is a stable price adjustment combination (Van der Laan and Talman, 1987; Herings, 1997).

EXAMPLE 4. Let (A) be the price adjustment process proposed in Kamiya (1990). Let (B) consist of assumptions on the economy guaranteeing that the excess demand function for each commodity is C^2 in the interior of $\mathbb{R}_{+}^{K}$, is continuous in $\mathbb{R}_{+}^{K} \setminus \{0\}$, and satisfies the boundary condition that $Z^k(p) > 0$ whenever $p \in \mathbb{R}_{+}^{K} \setminus \{0\}$ and $p^k = 0$. Let (C) be the condition of convergence starting from all points of $\mathbb{R}_{+}^{K}$, except for a set of measure zero. Then (A)–(B)–(C) is a stable price adjustment combination (Kamiya, 1990).

EXAMPLE 5. Suppose that the commodities are indivisible. Let (A) be the price adjustment process proposed in Gul and Stacchetti (2000). Let (B) include the assumption of gross substitutability. Let (C) be the condition of convergence via increasing prices, starting from any price vector with positive excess demand in all commodities. Then (A)–(B)–(C) is a stable price adjustment combination (Gul and Stacchetti, 2000).

The following theorem demonstrates that classical results on stability of competitive equilibrium can be imported to the analysis of the new auction procedure. We have:

THEOREM 2. *Suppose that the triplet* (A)–(B)–(C) *is a stable price adjustment combination for competitive economies. Consider the auction game where price adjustment is governed by process* (A) *and bidders have pure private values satisfying assumptions* (B). *Then, for initial prices* $p(0)$ *in accord with* (C), *and if participation in the auction is mandatory:*

(i) *sincere bidding by every bidder is an equilibrium of the auction game;*[6]

(ii) *with sincere bidding, the price vector converges to a Walrasian equilibrium price; and*

(iii) *with sincere bidding, the outcome is that of a Vickrey auction with reserve price of* $p(0)$.

[6] With judicious choice of the measurable selection from $Q_i(\bullet)$—see footnote 4.

PROOF. Suppose that all bidders $j \neq i$ bid sincerely in the auction game where price adjustment is governed by process (A), and consider any initial price $p(0)$ in accord with condition (C). One available strategy for bidder i is to also bid sincerely. Since (A)–(B)–(C) is a stable price adjustment combination, price then converges to a Walrasian price vector w. Thus, $w \in \mathcal{P}_i$, the set of all prices attainable by bidder i. The second sentence of the statement of Theorem 1 then is applicable, implying that sincere bidding is a best response by bidder i, and proving the theorem. ■

Since Examples 1–5 each provide examples of stable price adjustment combinations, then in light of Theorem 2, each except possibly Example 2 provides an environment where sincere bidding is an equilibrium of the new auction,[7] implying final price vectors that are Walrasian and outcomes from Vickrey auctions with reserve prices. We have:

COROLLARIES 1–4. *Suppose that the adjustment process, assumptions on bidder preferences, and conditions on the initial prices are taken from Examples* 1 *and* 3–5, *respectively. Then sincere bidding by every bidder is an equilibrium, implying convergence to a Walrasian price vector and the outcome of a Vickrey auction with a reserve price.*

6. RELATIONSHIP WITH THE VICKREY AUCTION

In Theorem 1, each bidder i received her payoff from the Vickrey auction with reserve price of $p(0)$. This payoff coincides with the payoff from the Vickrey auction if the starting price $p(0)$ is chosen appropriately. We have:

THEOREM 3. *If the initial price* $p(0)$ *is chosen such that the market clears without bidder i at* $p(0)$ *(i.e.,* $\sum_{j \neq i} q_j(p(0)) = S$*), if each bidder* $j \neq i$ *bids sincerely, if bidder i's bidding is constrained so as to generate piecewise smooth price paths from* $[0,T]$ *to* $\mathbb{R}^K_{++}$*, and if a Walrasian price vector w is attainable by bidder i (i.e., if* $w \in \mathcal{P}_i$*), then bidder i maximizes her payoff by selecting a Walrasian price vector and thereby receives exactly her Vickrey auction payoff.*

PROOF. Following the proof of Theorem 1, observe that if the initial price $p(0)$ is chosen such that the market clears without bidder i at $p(0)$, then the term $p(0) \cdot [S - q_{-i}(p(0))]$ that appears in Expression (12) equals zero. Applying Theorem 1, the remaining payoff term, when maximized, is exactly bidder i's payoff from the Vickrey auction. ■

[7] In Example 2, convergence is only guaranteed from boundary points (except for a set of measure zero), whereas Lemma 2 *et seq*. only treat price paths that lie entirely in the interior. However, work subsequent to Smale (1976) may remedy the gap in this reasoning.

While the hypothesis of Theorem 3 is easy to satisfy if all bidders are identical, it obviously is not generally possible to select an initial price $p(0)$ such that every bidder receives her payoff from a Vickrey auction. However, Theorem 3 suggests a more intricate, $(n+1)$-step procedure that could be followed so that *every* bidder receives a Vickrey payoff.

Begin with any initial price $p(0)$ in accord with condition (C). We first perform the following n steps (which may be done in any order):

Step 1: Run the auction procedure of naming a price $p(t)$, allowing bidders $j \neq 1$ to respond with quantities $x_j(t)$ while imposing $x_1(t) = 0$, and adjusting price according to adjustment process (A) until such price p_{-1} is reached that the market clears (absent bidder 1).

…

Step n: Run the auction procedure of naming a price $p(t)$, allowing bidders $j \neq n$ to respond with quantities $x_j(t)$ while imposing $x_n(t) = 0$, and adjusting price according to adjustment process (A) until such price p_{-n} is reached that the market clears (absent bidder n).

Then, the auction continues with an $(n+1)^{st}$ step, in which the auction *starts from price* p_{-n} (or any of the other generated prices p_{-i}) and is allowed to run normally:

Step $n+1$: Run the auction procedure of naming a price $p(t)$, allowing all bidders $i = 1, \ldots, n$ to respond with quantities $x_i(t)$, and adjusting price according to adjustment process (A) until such price w is reached that the market clears (with all bidders).

Finally, payoffs are computed as follows. The payment of bidder n is simply the result of the crediting formula of Eq. (7), calculated along the path from p_{-n} to w (that was generated in Step $n+1$). The payment of bidder i ($1 \leq i \leq n-1$) is also the result of the crediting formula of Eq. (7), but is now calculated along the union of *three* paths: the path from p_{-i} to $p(0)$ (that was generated in Step i); the path from p(0) to p_{-n} (that was generated in Step n); and the path from p_{-n} to w (that was generated in Step $n+1$). We have:

THEOREM 4. *Suppose that the triplet* (A)–(B)–(C) *is a stable price adjustment combination for competitive economies. Consider the* $(n+1)$-*step auction game where price adjustment is governed by process* (A) *and bidders have pure private values satisfying assumptions* (B). *Then, for initial prices* $p(0)$ *in accord with* (C), *sincere bidding by every bidder is an equilibrium of the* $(n+1)$-*step auction game, the price vector converges to a Walrasian equilibrium price, and the outcome is exactly that of a Vickrey auction.*

PROOF. By Lemma 2, the payment of each bidder i ($1 \le i \le n$) is independent of the path from p_{-i} to w. By Theorem 2, w corresponds to a Walrasian equilibrium and price converges to w. Furthermore, as in the proof of Theorem 3, the term $p(0) \cdot [S - x_{-i}(0)]$ that appears in Expression (12) equals zero. Thus, the remaining payoff term, when maximized, is exactly bidder i's payoff from the Vickrey auction. ■

With the assumptions restricted to those of Example 1 (gross substitutability), it is easy to make a comparison with the preceding literature. Select any $p(0)$ that has the property that—with any one bidder removed—there is still excess demand for every commodity, i.e.,

$$\sum_{j \neq i} q_j^k(p(0)) > S^k \text{ , for all } i \text{ and all } k. \tag{14}$$

Then, starting from any initial price $p(0)$ satisfying Eq. (14), each of the n price paths generated by Steps 1 through n, respectively, are increasing, as is the price path generated by Step n+1. Thus, by following n successive ascending auction trajectories, it is possible to replicate the outcome of the Vickrey auction.

7. CONCLUSION

The main results of this paper may superficially appear to be in conflict with the recent work of Gul and Stacchetti (2000). These authors conclude their Introduction (p. 69) by observing:

> "More importantly, we show that no dynamic auction can reveal sufficient information to implement the Vickrey mechanism if all Gross Substitutes preferences are allowed. Thus, the unit demand case of Demange *et al.* [1986] and the multiple homogeneous goods case of Ausubel [1997] are the most general environments for which generalizations of the English auction can be used to implement efficient, strategy-proof allocations."

By contrast, Theorems 2 and 4, above, establish dynamic auction procedures for strategic bidders with multi-unit demands for heterogeneous commodities such that the price vector converges to a Walrasian equilibrium price and the allocation is efficient. Corollary 1, above, treats the case of gross substitutability. However, the precise statement of Gul and Stacchetti's Theorem 6 only excludes implementing *literally* the Vickrey outcome by an auction in which price traces a *single* ascending trajectory. Since Corollary 1 of the current paper yields a Vickrey auction with a reserve price, while Theorem 4 of the current paper potentially requires the use of n distinct ascending trajectories to achieve literally the Vickrey outcome, the conflict herein is only with Gul and Stacchetti's interpretation—not with their literal theorem.

One immediate question that may be asked is whether the simple auction design for K heterogeneous commodities proposed in this paper can be complicated sufficiently to treat the case where bidders have interdependent values. Perry and Reny (1999b) provide an affirmative answer to this question for the case of homogeneous objects. They consider a model where each bidder receives a one-dimensional signal, and where each bidder's valuation depends on the signals received by all *n* bidders. They show: (1) my efficient ascending-bid auction for homogeneous goods leads to efficient outcomes with interdependent values and *two* bidders; and (2) by allowing bidders to submit *directed demands* (one against each other bidder), it is possible to obtain efficient outcomes with interdependent values and *n* bidders.

I now conjecture that essentially the same two steps can be replicated for the efficient ascending-bid auction for heterogeneous commodities proposed in the current paper. That is, with interdependent values and two bidders, my efficient ascending-bid auction for heterogeneous commodities should also lead to efficient outcomes; and, again, by allowing bidders to submit directed demands (one against each other bidder), it should be possible to obtain efficient outcomes with interdependent values and *n* bidders.

At the same time, such a complication of the current auction design is not entirely in the spirit of the current paper. The primary objective of the current design (as well as the predecessor design for homogeneous goods) was really to introduce efficient auction procedures sufficiently simple and practicable that they might actually find themselves adopted into widespread use someday. For the case of *K* heterogeneous commodities, the evident simplicity of the current design for bidders with pure private values should be apparent. A full Vickrey auction requires bidders to report their utilities over the entire *K*-dimensional space of quantity vectors. The current design only requires bidders to evaluate their demands along a one-dimensional path of price vectors. The current design thus seems more manageable for bidders and, therefore, more reasonable to attempt to bring into usage.

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